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Correlation at Low Temperature: II. Asymptotics

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The present paper is a continuation of ref. 4, where the truncated two-point correlation function for a class of lattice spin systems was proved to have exponential decay at low temperature, under a weak coupling assumption. In this paper we compute the asymptotics of the correlation function as the temperature goes to zero. This paper thus extends ref. 3 in two directions: The Hamiltonian function is allowed to have several local minima other than a unique global minimum, and we do not require translation invariance of the Hamiltonian function. We are in particular able to handle spin systems on a general lattice.

KEY WORDS: Spin systems; Witten Laplacian; Lattice Green's function.

1. INTRODUCTION AND RESULTS

1.1. Introduction

Let Λ be a finite set. The reader should think of Λ as an element of an infinite family of sets $\Gamma = \{\Lambda\}$ ordered by inclusion. Constants (real and positive) appearing in this paper which neither depend on a particular point $i \in \Lambda$ nor on the choice for $\Lambda \in \Gamma$ are called *universal*.

Given a Hamiltonian function $H = H_\Lambda \colon \mathbb{R}^\Lambda \to \mathbb{R}$, we define the associated Gibbs measure at inverse temperature $\beta > 0$ by

$$d\mu_\Lambda^\beta(x) := e^{-2\beta H(x)} \frac{d^\Lambda x}{\mathscr{Z}_\beta}. \tag{1.1}$$

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Here $x=(x_i)_{i\in\Lambda}\in\mathbb{R}^\Lambda:=\bigoplus_{i\in\Lambda}\mathbb{R}=\{x\colon\Lambda\to\mathbb{R}\}$, and $d^\Lambda x$ is the Lebesgue measure on $\mathbb{R}^\Lambda$. The constant $\mathscr{Z}_\beta=\int_{\mathbb{R}^\Lambda}e^{-2\beta H(x)}\,d^\Lambda x$ is a normalization constant chosen so that $d\mu_\Lambda^\beta$ is a probability measure. We assume the Hamiltonian H to be a sum of single-spin potentials $\{f_i\}_{i\in\Lambda}$ and pair-interactions $\{w_{ij}\}_{i,j\in\Lambda}$ of the form

$$H(x)=\sum_{i\in\Lambda}f_i(x_i)+\alpha\sum_{i,j\in\Lambda,i\neq j}w_{ij}(x_i,x_j).\tag{1.2}$$

We stress that the f_i's and $w_{ij}=w_{ji}$'s may have an explicit Λ-dependence. The coupling constant α is assumed to be small and positive. The f_i's should have a unique global minimum at $x_i=0$ and the interaction term should be ferromagnetic at 0. See Hypotheses 1, 2, and 3 below for a precise formulation.

The object to be studied in this paper is the truncated two-point correlation function given by

$$\mathbb{E}_\beta^T(x_i;x_j):=\mathbb{E}_\beta(x_ix_j)-\mathbb{E}_\beta(x_i)\,\mathbb{E}_\beta(x_j).\tag{1.3}$$

Here $\mathbb{E}_\beta(\cdot)$ denotes expectation value with respect to the Gibbs measure defined in (1.1), $\mathbb{E}_\beta(u):=\int_{\mathbb{R}^\Lambda}u\,d\mu_\Lambda^\beta$, for a polynomially bounded observable $u\colon\mathbb{R}^\Lambda\to\mathbb{R}$. In ref. 4 we assumed the interactions to decay exponentially fast. More precisely, we assumed the existence of a (universal) metric ρ on Λ such that w_{ij} is bounded by $e^{-\rho(i,j)}$, in a suitable sense. Under these assumptions, the following was shown, cf. 4, Theorem 1.1: For any $\varepsilon>0$ there exist universal constants α_0 and β_0 such that for $|\alpha|<\alpha_0$ and $\beta>\beta_0$, the correlations are bounded by

$$|\mathbb{E}_\beta^T(x_i\,;x_j)|\leqslant\frac{1+C\varepsilon}{2\beta\lambda_{\min}}\,e^{-(1-\varepsilon)\,\rho(i,j)},\tag{1.4}$$

where $\lambda_{\min}=\inf\sigma[H''(0)]>1/C>0$ is the lowest eigenvalue of the Hessian at $x=0$, and C is a universal constant.

The purpose of the present paper is to sharpen this result, in particular, to give upper *and* lower bounds on the correlations that agree asymptotically in the low temperature limit $\beta\to\infty$. To this end, we replace the metric ρ, which in ref. 4 was assumed to be given a priori, by ρ_H, which is essentially determined by the logarithm of the resolvent of the Hessian of H at $x=0$, i.e., $\rho_H(i,j)=\ln\{H''(0)^{-1}\}_{ii}+\ln\{H''(0)^{-1}\}_{jj}-\ln\{H''(0)\}_{ij}$, see (1.17). That is, all decay properties derived in this paper are to be compared to the decay of $H''(0)^{-1}$. Under a finite range assumption and a ferro-magnetic assumption on the interaction, but no assumption of translation invariance or any other geometric structure of the lattice, weimprove in the present paper the estimate (1.4) to the following: There exist universal constants α_0, β_0, and C such that for $0 \leqslant \alpha < \alpha_0$ and $\beta > \beta_0$ we have

$$\frac{2[1-\tau(\beta)]}{\beta}\{H''(0)^{-1}\}_{ij}^{1+\tau(\beta)} \leqslant \mathbb{E}_\beta^T(x_i\,;x_j) \leqslant \frac{2[1+\tau(\beta)]}{\beta}\{H''(0)^{-1}\}_{ij}^{1-\tau(\beta)}, \quad (1.5)$$

where $\tau(\beta) = C\beta^{-1/2}$. The precise formulation of this main result is given in Theorem 1.1 below. The correlation asymptotics have been derived in a form similar to (1.5) in ref. 3. The assumptions in ref. 3 were, however, more stringent than those used here (namely, the f_i's were forced to have only one critical point) and ruled out various important natural examples for H, like an Ising-ferromagnet in a uniform, non-zero, external magnetic field. In view of deriving the correlation asymptotics, rather than mere exponential bounds of type (1.4), the assumption of translation invariance was crucial in ref. 3, while no such requirement is necessary in the present paper. In fact, one of the novelties of our approach is based on the observation, that the assumption of existence of an a-priori metric is obsolete because the Hessian $H''(0)$ of H at $x = 0$ *defines* a metric ρ_H on Λ which yields the correlation length, see Theorem 1.1.

We approach the problem via a representation formula, see Theorem 2.6, which expresses the truncated two-point correlations functions in terms of matrix elements of the resolvent of a so-called Witten Laplacian (restricted to 1-forms). See Section 2. For a more thorough discussion of the Witten Laplacian techniques used here, and of related works, we refer the reader to the introduction to our first paper on the subject.[4] In the remaining part of this subsection we mention some recent works and one application which was not discussed in our previous paper.

A number of works, starting with a paper by Naddaf and Spencer,[13] uses semiclassical analysis of the Witten Laplacian on 1-forms to construct the continuum limit of some massless spin models. Here the lattice spacing plays the role of a semiclassical parameter. We refer the reader to the recent paper by Conlon,[6] and references therein, for further material related to this approach.

In a work[12] of Matte and the second author, the main technical form bound of ref. 3 is used to show that the usual semiclassical picture, of the low-lying spectrum of a Schrödinger operator with convex potential, persists in the thermodynamic limit.

After a ground state transform the Witten Laplacian on 0-forms takes the form $-\beta^{-2}\Delta + \nabla H \cdot \nabla$ as an operator on $L^2(\mathbb{R}^\Lambda; \exp(-\beta H)\, d^\Lambda x)$. In this form it appears often in the theory of kinetic equations, and was studied by Hérau and Nier who obtained bounds on the rate of convergence to equilibrium for the Fokker–Planck equation. See ref. 10, and references therein. In this connection no uniformity in Λ is sought for.

Helffer and Nier have recently obtained in ref. 8 delicate conditions under which a Poincaré inequality holds for the Gibbs measure (1.1). They approach the problem by proving the stronger statement that the Witten Laplacian on 0-forms has compact resolvent. The Poincaré inequalities thus obtained are not uniform in the cardinality of Λ.

1.2. Hypotheses on the Hamiltonian

The remaining part of this section is devoted to a presentation of the main result, Theorem 1.1, below. We begin by formulating the hypotheses under which we work. We start with the hypothesis on the self-energies f_j, which appear in the Hamiltonian (1.2).

Hypothesis 1. For any $j \in \Lambda$, zero is the unique, non-degenerate minimum of $f_j \in C^2(\mathbb{R}; \mathbb{R})$, attained at $t=0$, i.e., $f_j(0) = f'_j(0) = 0$, $f''_j(0) > 0$, and $f_j(t) > 0$, whenever $t \neq 0$. Moreover, there exist universal constants $0 < c_f \leqslant 1 \leqslant C_f$ and R_f, such that

$$c_f \leqslant f''_j(0) \leqslant C_f, \tag{1.6}$$

$$\forall t \neq 0: \quad f'_j(t) = 0 \Rightarrow f_j(t) \geqslant c_f, \tag{1.7}$$

$$\forall t \in \mathbb{R}: \quad |f''_j(t) - f''_j(0)| \leqslant C_f(|f'_j(t)| + \min\{1, |t|\}), \tag{1.8}$$

$$\forall\, |t| \geqslant R_f: \quad |f'_j(t)| \geqslant c_f \max\{|f'_k(s)| \mid |s| \leqslant |t|, k \in \Lambda\} \tag{1.9}$$

for all $j \in \Lambda$.

Condition (1.9) implies that the f_j's are monotonely increasing outside a ball of radius R_f, and that the slope of an f_j at a point t ($|t| > R_f$) dominates the slope, uniformly in Λ, of all the f_i's inside the ball of radius $|t|$. For the formulation of the hypotheses on the interactions w_{ij}, where $i \neq j$, it is convenient to use the following notation for the partial derivatives of w_{ij},

$$\partial_1 w_{ij}(x_i, x_j) := \frac{\partial w_{ij}}{\partial x_i}(x_i, x_j), \qquad \partial_2 w_{ij}(x_i, x_j) := \frac{\partial w_{ij}}{\partial x_j}(x_i, x_j),$$

$$\partial_1^2 w_{ij}(x_i, x_j) := \frac{\partial^2 w_{ij}}{\partial x_i^2}(x_i, x_j), \qquad \partial_2^2 w_{ij}(x_i, x_j) := \frac{\partial^2 w_{ij}}{\partial x_j^2}(x_i, x_j), \qquad \text{and}$$

$$\partial_{12}^2 w_{ij}(x_i, x_j) := \frac{\partial^2 w_{ij}}{\partial x_i\, \partial x_j}(x_i, x_j). \tag{1.10}$$

We introduce two symmetric matrices $\underline{a} = (a_{ij})_{i,\, j \in \Lambda}$, $\underline{s} = (s_{ij})_{i,\, j \in \Lambda}$ by

$$a_{ij} := -\partial_{12}^2 w_{ij}(0, 0) \qquad \text{and} \qquad s_{ij} := \begin{cases} 1, & \text{if} \quad a_{ij} \neq 0 \\ 0, & \text{if} \quad a_{ij} = 0. \end{cases} \tag{1.11}$$

Hypothesis 2. There exist universal constants $C_s \geqslant 1$ and C_a such that the two symmetric matrices $\underline{a} = (a_{ij})_{i,\, j \in \Lambda}$ and $\underline{s} = (s_{ij})_{i,\, j \in \Lambda}$ possess the following properties:

$$\forall i, j \in \Lambda: \quad a_{ij} \geqslant 0, \tag{1.12}$$

$$\max_{i \in \Lambda} \sum_{j \in \Lambda} a_{ij} \leqslant C_a \qquad \text{and} \qquad \max_{i \in \Lambda} \sum_{j \in \Lambda} s_{ij} \leqslant C_s. \tag{1.13}$$

We remark that (1.12) is a ferromagnetic property. The second bound in Eq. (1.13) can be viewed as a finite-range condition on the Hessian of H at $x = 0$. Namely, given $i \in \Lambda$, the number of nearest-neighbor sites of i is $\sum_{j \in \Lambda} s_{ij}$. Condition (1.13) requires these to be uniformly bounded in $i \in \Lambda$.

The second hypothesis on the interactions w_{ij} is formulated with the aid of the following functions,

$$h_j(s) := \min\{|f'_j(s)|, |f'_j(s)|^{\frac{1}{2}}\} + \min\{1, |s|\}, \tag{1.14}$$

which we introduce for all $j \in \Lambda$.

Hypothesis 3. For all $i, j \in \Lambda$, the pair interaction functions $w_{ij} = w_{ji} \in C^2(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ vanish on-site and at the origin, i.e., $w_{ii}(s, t) \equiv 0$ and $w_{ij}(0, 0) = 0$. Furthermore, for $i, j \in \Lambda$, there exists a universal number C_w, such that

$$\begin{aligned} &|\partial_1 w_{ij}(x_i, x_j)| + |\partial_2 w_{ij}(x_i, x_j)| + |\partial_1^2 w_{ij}(x_i, x_j) - \partial_1^2 w_{ij}(0, 0)| \\ &\qquad + |\partial_2^2 w_{ij}(x_i, x_j) - \partial_2^2 w_{ij}(0, 0)| + |\partial_{12}^2 w_{ij}(x_i, x_j) - \partial_{12}^2 w_{ij}(0, 0)| \\ &\quad \leqslant C_w a_{ij}(h_i(x_i) + h_j(x_j)), \end{aligned} \tag{1.15}$$

and

$$|\partial_1^2 w_{ij}(0, 0)| + |\partial_2^2 w_{ij}(0, 0)| \leqslant C_w a_{ij}. \tag{1.16}$$

All (derived) universal constants appearing in the paper depend only on ingredients through the universal constants which appear in the relevant hypotheses, i.e., c_f, C_f, R_f, C_a, C_s, and C_w from Hypotheses 1, 2, and 3.

By ref. 4, Lemma B.1, Hypotheses 1, 2, and 3 insure the well-definedness of the Gibbs measure (1.1), for small α. Moreover, polynomially bounded, measurable observables $u\colon \mathbb{R}^{\Lambda} \to \mathbb{C}$ are integrable.

Example. Before stating the main result, we pause to consider a simple class of examples on the cubic lattice $\Lambda \subset \mathbb{Z}^d$, which satisfies Hypothesis 1, 2, and 3. Let $f_i(t) = p(t)$, where p is a polynomial of even degree (at least 2). In particular $p(t) = t^4 - t^2 + ht + c_h$ is of this type, where $h \neq 0$ and c_h is chosen such that $\min_t p(t) = 0$. As for the interaction we take nearest neighbour interaction on Λ, i.e., $w_{ij} = 0$, if $|i-j|_1 \neq 1$. Hypothesis 3 is fulfilled if $w_{ij} \in C^3(\mathbb{R} \times \mathbb{R}; \mathbb{R})$, $w_{ij}(0,0) = 0$, and $|\partial^\alpha w_{ij}(s,t)|$ are uniformly bounded in $(s,t) \in \mathbb{R}^2$, for $|\alpha| \in \{2,3\}$. In addition $\partial_{12}^2 w_{ij}(0,0) \leqslant 0$ is required for Hypothesis 2. In particular $\omega_{ij}(s,t) = -st$ is of this type.

1.3. The Main Result on Correlation Asymptotics

A Neumann series expansion shows that, for $\alpha \geqslant 0$ small enough, we have $\{H''(0)^{-1}\}_{ij} \geqslant 0$, and in particular $\{H''(0)^{-1}\}_{jj} > 0$ (see first paragraph in Sec. 3). We may hence define a map $\rho_H\colon \Lambda \times \Lambda \to [0,\infty]$ by

$$\exp[-\rho_H(i,j)] := \frac{\{H''(0)^{-1}\}_{ij}}{\{H''(0)^{-1}\}_{ii}^{1/2}\,\{H''(0)^{-1}\}_{jj}^{1/2}}. \tag{1.17}$$

We are now in a position to formulate the main result of this paper.

Theorem 1.1. Assume Hypotheses 1, 2, and 3. There exist universal constants C, β_0, and $\alpha_0 > 0$ such that, for any $0 \leqslant \alpha \leqslant \alpha_0$ and any $\beta \geqslant \beta_0$, we have

$$\frac{2\,[1-\tau(\beta)]}{\beta}\,\{H''(0)^{-1}\}_{ij}^{1+\tau(\beta)} \leqslant \mathbb{E}_\beta^T(x_i; x_j) \leqslant \frac{2\,[1+\tau(\beta)]}{\beta}\,\{H''(0)^{-1}\}_{ij}^{1-\tau(\beta)}, \tag{1.18}$$

where $\tau(\beta) := C\beta^{-1/2}$. Moreover, $\rho_H\colon \Lambda \times \Lambda \to [0,\infty]$ defined by (1.17) is a metric on Λ, and

$$\left|\frac{\ln\,[\frac{\beta}{2}\,\{H''(0)^{-1}\}_{ii}^{-1/2}\,\mathbb{E}_\beta^T(x_i; x_j)\{H''(0)^{-1}\}_{jj}^{-1/2}] + \rho_H(i,j)}{1+\rho_H(i,j)}\right| \leqslant \tau(\beta). \tag{1.19}$$

We stress that we do not impose any translation invariance assumption on the Hamiltonian, which was crucial for the method used in refs. 3, 14, and 15.

Theorem 1.1 reduces the problem of studying the correlation function at low temperature to that of the study of resolvents of transition matrices. This can be viewed as a problem related to random walks on infinite graphs, or more precisely, Ornstein–Zernike theory. We refer the interested reader to Sjöstrand,[14] Section 5, Campanino, Ioffe, and Velenik[5] (and references therein), and the monographs by Spitzer[16] (translation invariant random walks on Z^d) and Woess[17] (general theory).

2. A RECOLLECTION OF EARLIER RESULTS

In this Section we recall the ingredients and results from ref. 4 which are used here. They hold under Hypotheses 1, 2, 3, and a seemingly additional requirement of existence of an a-priori metric ρ on Λ satisfying the summability condition

$$\max_i \sum_j e^{-\rho(i,j)} \leqslant C_\rho \qquad \text{and} \qquad \max_i \sum_j e^{\rho(i,j)} a_{ij} \leqslant C_\rho, \tag{2.1}$$

for a universal constant C_ρ. For the construction of ρ from Hypothesis 2, we introduce the set of *nearest-neighbour bonds*

$$\mathscr{B}_a := \{(i,j) \in \Lambda \times \Lambda \mid a_{ij} \neq 0\} = \{(i,j) \in \Lambda \times \Lambda \mid s_{ij} = 1\}. \tag{2.2}$$

Given two points $i, j \in \Lambda$, a nonempty, finite collection of nearest-neighbour bonds of the form $\gamma = \{(i_0, i_1), (i_1, i_2),..., (i_{n-1}, i_n)\} \subseteq \mathscr{B}_a$, with $i_0 = i$, $i_n = j$, and $i_k \neq i_{k+1}$, is called a *path* from i to j and is denoted $\gamma: i \to j$. The number $|\gamma| := n$ of bonds $b = (i_k, i_{k+1}) \in \mathscr{B}_a$ in the path is referred to as its *length*. The collection of all paths from i to j is denoted $\Gamma(i, j)$. Note that (i, i) is not a nearest-neighbour bond (i.e., $(i, i) \notin \mathscr{B}_a$, which follows from $a_{ii} = 0$). Furthermore, neither $\varnothing$ nor $\{(i, i)\}$ are paths. A metric d is defined as the canonical metric of the graph $(\Lambda, \mathscr{B}_a)$. So, given two points $i, j \in \Lambda$, their distance with respect to the metric d is defined to be the minimal length of all paths linking i and j, i.e.,

$$d(i,j) := \min\{|\gamma| \mid \gamma \in \Gamma(i,j)\}, \tag{2.3}$$

$d(i, i) := 0$, and $d(i, j) := \infty$, if no such path exists. If $d(i, j) = 1$ then $(i, j) \in \mathscr{B}_a$ and i and j are called *nearest neighbours*. We note that $\{H''(0)^{-1}\}_{i,j} = 0$, i.e., $\rho_H(i, j) = \infty$, if and only if i and j are not in the same connected component of Λ, with respect to the metric d.

The following lemma states that a sufficiently large multiple of d satisfies the summability condition required in ref. 4.

Lemma 2.1. Assume Hypotheses 1, 2, 3, and define a metric $\rho := \ln(2C_s)\, d$ on Λ. Then

$$\max_i \sum_{j(\neq i)} e^{-\rho(i,j)} \leqslant 1 \qquad \text{and} \qquad \max_i \sum_j e^{\rho(i,j)}\, a_{ij} \leqslant 2C_s C_a. \tag{2.4}$$

Proof. To derive the first estimate in (2.4), we remark that, given a fixed site $i \in \Lambda$, the number of sites $j \in \Lambda$, which are at distance $n = d(i, j)$ of i, is bounded by C_s^n, due to Hypothesis 2. Therefore,

$$\sum_{j(\neq i)} e^{-\rho(i,j)} \leqslant \sum_{n=1}^{\infty} C_s^n e^{-n \ln(2C_s)} = 1. \tag{2.5}$$

The second estimate in (2.4) is a trivial consequence of Hypothesis 2 which implies that $d(i, j) = 1$, whenever $a_{ij} > 0$. ▮

2.1. Modified Single-Spin Potentials

We first introduce modifications g_i's of the f_i's, which coincide with the f_i's near 0 and, as the main point, differ from the f_i's by having no local minima away from 0. They were constructed in ref. 4, Lemma 1.2, and we present them in the following lemma, leaving out those properties not needed here. The reader might find Fig. 1 instructive.

Lemma 2.2. Assume Hypothesis 1. There exist universal numbers $q_{\max} > 0$ and $0 < \hat{R}_0 < \hat{R}_1 < \check{R}_0 < \check{R}_1$, to which we associate the unions of intervals

$$\begin{aligned} I_0 &:= [-\hat{R}_0, \hat{R}_0], & \hat{I} &:= (-\hat{R}_1, -\hat{R}_0) \cup (\hat{R}_0, \hat{R}_1), \\ \check{I} &:= (-\infty, -\check{R}_0) \cup (\check{R}_0, \infty), & I_\infty &:= (-\infty, -\check{R}_1) \cup (\check{R}_1, \infty), \end{aligned} \tag{2.6}$$

and functions

$$\begin{aligned} &\hat{q}_j \in C^2(\hat{I}; [0, q_{\max}]), \qquad \check{q}_j \in C^2(\check{I}; [0, q_{\max}]), \\ &\text{and} \qquad q_j := \hat{q}_j + \check{q}_j \in C^2(\mathbb{R}, [0, q_{\max}]), \end{aligned} \tag{2.7}$$

possessing the following properties:

(i) On I_∞, we have $\check{q}_j \equiv q_{\max}$.

(ii) The functions

$$g_j := f_j - q_j \qquad \text{and} \qquad \hat{g}_j := f_j - \hat{q}_j \tag{2.8}$$

are nonnegative and have a unique critical point at $t = 0$.

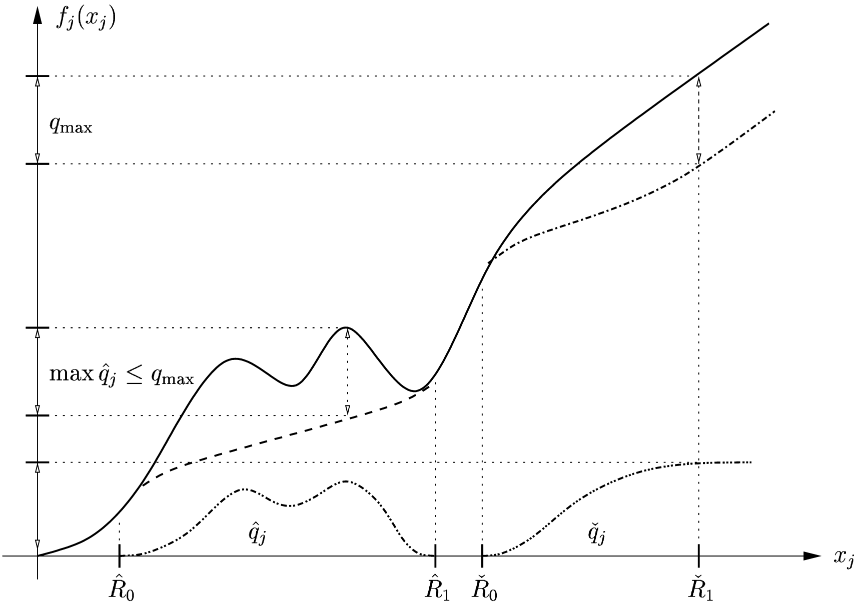

Fig. 1. The solid curve is the graph of f_j. The graph of $\hat{g}_j$ is the dashed curve on $(\hat{R}_0, \hat{R}_1)$ and agrees with f_j on $(0, \hat{R}_0)$ and on $(\hat{R}_1, \infty)$. The graph of g_j is the dashed curve on $(\hat{R}_0, \hat{R}_1)$, the dash-dotted curve on $(\check{R}_0, \infty)$, and agrees with f_j on $(0, \hat{R}_0)$ and on $(\hat{R}_1, \check{R}_0)$. The dash-dot-dotted curve depicts $\hat{q}_j$, and the dash-dot-dot-dotted curve depicts $\check{q}_j$

(iii) There exist universal constants $c_g, C_g > 0$ such that, $\sup_t |q_j'(t)| < C_g$, and for all $t \in \mathbb{R}$

$$c_g \min\{1, |t|\} \leqslant \operatorname{sgn}(t)\, g_j'(t) \leqslant C_g\, e^{C_g |t|}. \tag{2.9}$$

2.2. Semiclassical Localization Estimates

The second result we invoke in this paper is a semiclassical localization estimate deriving from ref. 4, Theorem 1.6, which we use to argue that the twists we introduce into the Witten Laplacian only give rise to small corrections in the low temperature limit. It holds under Hypotheses 1, 2, 3, and the existence of a metric d on Λ satisfying the summability condition (2.1), which is insured by Lemma 2.1.

Theorem 2.3. Assume Hypotheses 1, 2, and 3, and let $R > 0$ be a fixed universal number. Then there exist universal constants $\alpha_0, \beta_0, \delta > 0$, such that, for all $k \in \Lambda$, $0 \leqslant \alpha \leqslant \alpha_0$, and $\beta \geqslant \beta_0$, we have

$$\int_{|x_k| \geqslant R} e^{-2\beta H(x)} \frac{d^\Lambda x}{\mathscr{Z}_\beta} \leqslant e^{-\beta\,\delta}. \tag{2.10}$$

We remark that, for a given lattice site $k \in \Lambda$, the Hamiltonian function $\tilde{H}_k(x) := H(x) - q_k(x_k)$ also satisfies Hypotheses 1, 2, and 3, uniformly in k (i.e., with constants independent of k). Moreover, $H(x) \equiv \tilde{H}_k(x)$ on $\{x \in \mathbb{R}^\Lambda : |x_k| \leqslant \hat{R}_0\}$. Thus, an application of Theorem 2.3 to the metric d and the Hamiltonian function $\tilde{H}_k(x)$ yields the following corollary.

Corollary 2.4. Assume Hypotheses 1, 2, and 3. Then there exist universal constants $\alpha_0, \beta_0, \delta > 0$, such that, for all $k \in \Lambda$, $0 \leqslant \alpha \leqslant \alpha_0$, and $\beta \geqslant \beta_0$, we have

$$\left| \int e^{-2\beta[H(x) - q_k(x)]} \frac{d^\Lambda x}{\mathscr{Z}_\beta} - 1 \right| \leqslant e^{-\beta\delta}. \tag{2.11}$$

It is important to notice that Hypotheses 1, 2, and 3 and, hence, also Corollary 2.4 may fail to hold for the Hamiltonian function $H(x) - 2q_k(x_k)$. This fact plays a certain role for our choice of the projection p in Eq. (4.1).

2.3. Twisted de Rham Complex

The following is a brief recollection of the remaining part of ref. 4, Section 1, and we refer the reader to ref. 4, Section 1 and Appendix B for details. We use the summation convention $\sum_i(\cdot) := \sum_{i \in \Lambda}(\cdot)$, $\sum_{i,j}(\cdot) := \sum_{(i,j) \in \Lambda^2}(\cdot)$, $\sum_{i \neq j}(\cdot) := \sum_{(i,j) \in \Lambda^2 \setminus \{(i,i) \mid i \in \Lambda\}}(\cdot)$, and $\sum_{i(\neq j)}(\cdot) := \sum_{i \in \Lambda \setminus \{j\}}(\cdot)$. Next, we introduce the fermionic Fock space over $\mathbb{C}^\Lambda$,

$$\mathscr{F} \equiv \mathscr{F}(\mathbb{C}^\Lambda) := \bigoplus_{n=0}^{\infty} \mathscr{F}^{(n)}, \qquad \mathscr{F}^{(n)} := (\mathbb{C}^\Lambda)^{\otimes_a n}, \tag{2.12}$$

where $\otimes_a n$ denotes the n-fold antisymmetric tensor product, and $\mathscr{F}^{(0)} \simeq \mathbb{C}\Omega$ is a one-dimensional subspace spanned by the normalized vacuum vector Ω. The standard annihilation and creation operators $\{a_i, a_i^*\}_{i \in \Lambda}$ represent the canonical anticommutation relations (CAR); $\forall i, j \in \Lambda$:

$$\{a_i, a_j\} = \{a_i^*, a_j^*\} = 0, \quad \{a_i^*, a_j\} = \delta_{ij}, \quad \text{and} \quad a_i \Omega = 0, \tag{2.13}$$

on $\mathscr{F}$, where $\{A, B\} = AB + BA$, and δ_{ij} is the Kronecker delta. The Hilbert space of forms over $\mathbb{R}^\Lambda$ is the tensor product

$$\mathscr{H} := L^2(\mathbb{R}^\Lambda) \otimes \mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{H}^{(n)}, \qquad \mathscr{H}^{(n)} = L^2(\mathbb{R}^\Lambda) \otimes \mathscr{F}^{(n)}. \tag{2.14}$$

We introduce a (multiple of the) standard exterior derivative on $\mathscr{H}$, for $\beta > 0$, and its adjoint

$$d := \sum_i \frac{1}{\beta} \partial_i \otimes a_i^* \qquad \text{and} \qquad d^* = -\sum_i \frac{1}{\beta} \partial_i \otimes a_i, \tag{2.15}$$

where ∂_i is shorthand for $\frac{\partial}{\partial x_i}$. Note that $d^2 = (d^*)^2 = 0$. The Hodge Laplacian associated to this exterior derivative is $dd^* + d^*d = (d + d^*)^2$.

To an operator T on $\mathscr{H}^{(1)}$, we associate its second quantization $d\Gamma(T)\colon \mathscr{H} \to \mathscr{H}$ by the standard formula, i.e., if T is represented by a matrix $(T_{ij})_{i,j \in \Lambda}$ whose entries take their values in operators on $L^2(\mathbb{R}^\Lambda)$ then

$$d\Gamma(T) := \sum_{i,j} T_{ij} \otimes a_i^* a_j. \tag{2.16}$$

In the present paper we second-quantize only operators whose entries are semi-bounded, self-adjoint operators on $L^2(\mathbb{R}^\Lambda)$.

We remark that, if $\bigcap_{i,j} \mathscr{D}(T_{ij})$ is a core for all the T_{ij}'s, then the operators T and $d\Gamma(T)$ are essentially self-adjoint and semi-bounded on $\{\bigcap_{i,j} \mathscr{D}(T_{ij})\} \otimes \mathbb{C}^\Lambda$ and $\{\bigcap_{i,j} \mathscr{D}(T_{ij})\} \otimes \mathscr{F}$ respectively. If T is bounded then $d\Gamma(T)$ is also bounded.

In particular, let Q, $\hat{Q}$, and $\check{Q}$ denote the matrix-valued functions with entries

$$\hat{Q}_{ij}(x) = \delta_{ij} \hat{q}_i(x_i), \qquad \check{Q}_{ij}(x) = \delta_{ij} \check{q}_i(x_i) \tag{2.17}$$

$$\text{and} \quad Q_{ij}(x) = \hat{Q}_{ij}(x) + \check{Q}_{ij}(x). \tag{2.18}$$

Here $x \equiv (x_i)_{i \in \Lambda} \in \mathbb{R}^\Lambda$, and $\hat{q}_i$ and $\check{q}_i$ are introduced in (2.7). We frequently omit the argument and write $\hat{q}_i := \hat{q}_i(x_i)$ and $\check{q}_i := \check{q}_i(x_i)$, etc. Then their second quantization is given by

$$d\Gamma(Q^\#) = \sum_j q_j^\# \otimes a_j^* a_j, \tag{2.19}$$

where $Q^\#$ denotes Q, $\hat{Q}$, or $\check{Q}$, and $q_j^\#$ denotes q_j, $\hat{q}_j$, or $\check{q}_j$, respectively.

Now, we introduce the twisted exterior derivative

$$d_{H,Q} := e^{-\beta(H - d\Gamma(Q))} \, d e^{\beta(H - d\Gamma(Q))} \tag{2.20}$$

on $C_0^\infty(\mathbb{R}^\Lambda) \otimes \mathscr{F}$, which is a core for $d_{H,Q}$. We denote its closure by the same symbol. Here $H \equiv H \otimes \mathbb{1}$ is considered a multiplication operator on $\mathscr{H}$. The *twisted Dirac operator* is the sum $d_{H,Q} + d_{H,Q}^*$ of the twisted exterior derivative and its adjoint. By construction it is clear that $d_{H,Q}^2 = (d_{H,Q}^*)^2 = 0$. Thus the square of the twisted Dirac operator is the associated Hodge Laplacian

$$\Delta_{H,Q} := (d_{H,Q} + d_{H,Q}^*)^2 = d_{H,Q}\, d_{H,Q}^* + d_{H,Q}^*\, d_{H,Q}, \tag{2.21}$$

which we call the *twisted Witten Laplacian*. Similarly to the situation in ref. 4, we have that $C_0^\infty(\mathbb{R}^\Lambda) \otimes \mathscr{F}$ is a formcore for $\Delta_{H,Q}$.

We write $\Delta_{H,Q}^{(n)}$ for the restriction of $\Delta_{H,Q}$ to $\mathscr{H}^{(n)}$. We recall (ref. 4, Theorem 1.4)

Theorem 2.5. Assume Hypotheses 1, 2, and 3.

(i) There exist $\alpha_0, \beta_0 > 0$ such that, for $0 \leqslant \alpha < \alpha_0$ and $\beta > \beta_0$,

$$\operatorname{Ker}\{\Delta_{H,Q}^{(0)}\} = \mathbb{C}\, e^{-\beta H} \qquad \text{and} \qquad \operatorname{Ker}\{\Delta_{H,Q}^{(1)}\} = \{0\}. \tag{2.22}$$

(ii) If all the f_i's and w_{ij}'s are C^∞ in (not necessarily universal) neighbourhoods of 0, then there exists a universal constant $\alpha_0 > 0$ such that for $0 \leqslant \alpha < \alpha_0$ and all $\beta > 0$,

$$\operatorname{Ker}\{\Delta_{H,Q}\} = \mathbb{C}e^{-\beta H} \otimes \Omega. \tag{2.23}$$

In ref. 4, the following representation of the correlations has been derived from Theorem 2.5, which is of key importance to our analysis.

Theorem 2.6. Assume Hypotheses 1, 2, and 3. There exist $\alpha_0, \beta_0 > 0$ such that, for $0 \leqslant \alpha < \alpha_0$ and $\beta > \beta_0$,

$$\mathbb{E}_\beta^T(x_i\,;x_j) = \frac{1}{\beta^2 \mathscr{Z}_\beta} \langle e^{-\beta(H-q_i)} \otimes e_i \,|\, (\Delta_{H,Q}^{(1)})^{-1}\, e^{-\beta(H-q_j)} \otimes e_j \rangle. \tag{2.24}$$

For $Q = 0$, (2.24) has first been observed (implicitly) by Helffer and Sjöstrand in ref. 9 and, more explicitly, by Sjöstrand in ref. 14. We have also the following important supersymmetric property.

Theorem 2.7. Assume Hypotheses 1, 2, and 3. There exist universal constants $\alpha_0, \beta_0 > 0$, such that, for $0 \leqslant \alpha < \alpha_0$ and $\beta > \beta_0$,

$$\sigma(\Delta_{H,Q}^{(0)}) \setminus \{0\} \subseteq \sigma(\Delta_{H,Q}^{(1)}). \tag{2.25}$$

Proof. The proof of Theorem 2.7 in ref. 14 assumes the discreteness of the spectrum of $\Delta_{H,Q}$. Here we refer instead to an abstract result of Johnsen, cf. ref. 11, Theorem 3.1. We apply this result to $H = \mathscr{H}^{(0)}$, $H_1 = \mathscr{H}^{(1)}$, $T = d_{H,Q\,|\,\mathscr{H}^{(0)}}$, and $F_1 = \overline{\operatorname{Ran} T}$. We verify one of the five equivalent conditions in Johnsen's theorem, namely condition (iv). It requires that $TT^*_{|F_1}$ has closed range and does not have 0 in its spectrum. But this follows directly from the estimate $TT^*_{|F_1} \geqslant \Delta^{(1)}_{H,Q\,|\,F_1}$ and (2.51) below. A similar argument is used in ref. 12 (for the case $Q = 0$). ∎

2.4. Explicit Expressions

We now give more explicit formulas for the objects introduced above. First, we introduce some exponential weights,

$$\hat{\Theta}_i := e^{\beta \hat{q}_i}, \qquad \check{\Theta}_i := e^{\beta \check{q}_i} \qquad \text{and} \qquad \Theta_i := e^{\beta q_i} = \hat{\Theta}_i\, \check{\Theta}_i, \tag{2.26}$$

and twisted derivatives, together with their adjoints,

$$Z_i(H) := \frac{1}{\beta}\, \partial_i + H'_i \qquad \text{and} \qquad Z_i^*(H) = -\frac{1}{\beta}\, \partial_i + H'_i. \tag{2.27}$$

We compute, using (2.13) and (2.15)–(2.20),

$$d_{H,Q} = \sum_i \Theta_i Z_i(H) \otimes a_i^* \qquad \text{and} \qquad d^*_{H,Q} = \sum_i Z_i^*(H)\, \Theta_i \otimes a_i. \tag{2.28}$$

Note the intertwining relations

$$\Theta_i Z_i(H) = Z_i(G)\, \Theta_i \qquad \text{and} \qquad \hat{\Theta}_i Z_i(H) = Z_i(\hat{G})\, \hat{\Theta}_i, \tag{2.29}$$

where

$$G := H - \sum_i q_i = \sum_i g_i + \alpha \sum_{i,j} w_{ij}, \tag{2.30}$$

$$\hat{G} := H - \sum_i \hat{q}_i = \sum_i \hat{g}_i + \alpha \sum_{i,j} w_{ij}. \tag{2.31}$$

The intertwining relations (2.29) were of key importance to our analysis in ref. 4, as they allowed us to pass from H to a new Hamiltonian function G which agrees with H at zero, but has yet no critical points other than zero.

Note that $\mathscr{H}^{(0)} = L^2(\mathbb{R}^\Lambda) \otimes \mathscr{F}^{(0)}$ can be identified with $L^2(\mathbb{R}^\Lambda)$, and $\mathscr{H}^{(1)}$ with $L^2(\mathbb{R}^\Lambda) \otimes \mathbb{C}^\Lambda$. We will use the same notation, $\Delta^{(0)}_{H,Q}$ and $\Delta^{(1)}_{H,Q}$, for both representations.

The fully twisted Witten Laplacian is of the form (cf. (2.13), (2.21), (2.28), and (2.29))

$$\begin{aligned}\Delta_{H,Q} &= \sum_{i,j} \Theta_i Z_i(H)\, Z_j^*(H)\, \Theta_j \otimes a_i^* a_j + Z_j^*(H)\, \Theta_j \Theta_i Z_i(H) \otimes a_j a_i^* \\ &= \sum_j \{\Theta_j Z_j(H)\, Z_j^*(H)\, \Theta_j \otimes a_j^* a_j + \Theta_j Z_j^*(G)\, Z_j(G)\, \Theta_j \otimes a_j a_j^*\} \\ &\quad + \sum_{i \neq j} [\Theta_i Z_i(H),\ Z_j^*(H)\, \Theta_j] \otimes a_i^* a_j, \\ &= \sum_j \{\Theta_j Z_j(H)\, Z_j^*(H)\, \Theta_j \otimes a_j^* a_j + \Theta_j Z_j^*(G)\, Z_j(G)\, \Theta_j \otimes a_j a_j^*\} \\ &\quad + \frac{2}{\beta} \sum_{i \neq j} \Theta_i \Theta_j H''_{ij}(x) \otimes a_i^* a_j,\end{aligned} \tag{2.32}$$

where we used that $[Z_i(H), \Theta_j] = 0$ and $[Z_i(H), Z_j^*(H)] = 2\beta^{-1} H''_{ij}$, for $i \neq j$, and we denote $H''_{ij} := \partial_i \partial_j H$. Restricting (2.32) to $\mathscr{H}^{(0)}$ and $\mathscr{H}^{(1)}$, we arrive at the twisted Witten Laplacian on 0- and 1-forms,

$$\Delta^{(0)}_{H,Q} = \sum_i \Theta_i Z_i^*(G)\, Z_i(G)\, \Theta_i, \tag{2.33}$$

$$\begin{aligned}\Delta^{(1)}_{H,Q} &= \sum_j \left\{\Theta_j\, Z_j(H)\, Z_j^*(H)\, \Theta_j + \sum_{k(\neq j)} \Theta_k Z_k^*(G)\, Z_k(G)\, \Theta_k\right\} \otimes E_{jj} \\ &\quad + \frac{2}{\beta} \sum_{j \neq k} \Theta_j\, \Theta_k\, H''_{jk} \otimes E_{jk}.\end{aligned} \tag{2.34}$$

Here E_{jk} denotes the jk^{th} unit matrix, i.e., the matrix with entries $(E_{jk})_{i\ell} = \delta_{ij}\delta_{k\ell}$.

2.5. Comparison Operator and Perturbation

We now recall the definition of the comparison operator which was shown in ref. 4 to approximate $\Delta_{H,Q}$ at low temperature in a form sense

$$A_{H,Q} := \sum_j \{\check{A}_j \otimes a_j^*\, a_j + A_j \otimes a_j\, a_j^*\} + \frac{2}{\beta} \sum_{i,j} H''_{ij}(0) \otimes a_i^* a_j, \tag{2.35}$$

where

$$\check{A}_j := \Theta_j Z_j(H)\, Z_j^*(H)\, \Theta_j - 2\beta^{-1}\, \check{\Theta}_j^2\, \hat{G}''_{jj}(x), \tag{2.36}$$

$$A_j := \Theta_j Z_j^*(G)\, Z_j(G)\, \Theta_j. \tag{2.37}$$

Furthermore, we define

$$W_{H,Q}(x) := W_{\text{diag}}(x) + W_{\text{off-d}}(x), \tag{2.38}$$

$$W_{\text{diag}}(x) := \frac{2}{\beta} \sum_j (\check{\Theta}_j^2 \, \hat{G}''_{jj}(x) - H''_{jj}(0)) \otimes a_j^* a_j, \tag{2.39}$$

$$\begin{aligned} W_{\text{off-d}}(x) &:= \frac{2}{\beta} \sum_{i \neq j} (\Theta_i \Theta_j \, H''_{ij}(x) - H''_{ij}(0)) \otimes a_i^* a_j \\ &= \frac{2\,\alpha}{\beta} \sum_{i \neq j} (\Theta_i \Theta_j \, \partial_{12}^2 w_{ij}(x_i, x_j) - \partial_{12}^2 w_{ij}(0,0)) \otimes a_i^* a_j, \end{aligned} \tag{2.40}$$

and we observe the decomposition identity

$$\varDelta_{H,Q} = A_{H,Q} + W_{H,Q}. \tag{2.41}$$

The argument "x" in (2.38)–(2.40) indicates that these operators act as matrix-valued multiplication operators and contain no differential operator. We frequently omit to display x.

The restrictions of $A_{H,Q}$, $W_{H,Q}$, W_{diag}, and $W_{\text{off-d}}$ onto $\mathscr{H}^{(0)}$ are given by

$$A_{H,Q}^{(0)} = \sum_j A_j = \sum_j \Theta_j Z_j^*(G) \, Z_j(G) \, \Theta_j, \tag{2.42}$$

$$W_{H,Q}^{(0)} = W_{\text{diag}}^{(0)} = W_{\text{off-d}}^{(0)} = 0. \tag{2.43}$$

Before we write down the restrictions of $A_{H,Q}$, $W_{H,Q}$, W_{diag}, and $W_{\text{off-d}}$ onto $\mathscr{H}^{(1)} \simeq \mathscr{H}^{(0)} \otimes \mathbb{C}^{\varLambda}$, we note that we may view any operator on $\mathscr{H}^{(1)}$ as a $\varLambda \times \varLambda$-matrix with entries in the operators on $\mathscr{H}^{(0)}$. More specifically, given an operator X on $\mathscr{H}^{(1)}$, we denote by $(\{X\}_{ij})_{i,j \in \varLambda}$ the unique family of operators on $\mathscr{H}^{(0)}$ such that

$$X = \sum_{i,j} \{X\}_{ij} \otimes E_{ij}. \tag{2.44}$$

Equipped with this notation, we find

$$A_{H,Q}^{(1)} = \sum_j \tilde{A}_j \otimes E_{jj} + \frac{2}{\beta} \sum_{i,j} H''_{ij}(0) \otimes E_{ij}, \tag{2.45}$$

$$\tilde{A}_j := \check{A}_j + \sum_{k(\neq j)} A_k, \tag{2.46}$$

$$\{W_{H,Q}^{(1)}\}_{ij} = \delta_{ij} \{W_{\text{diag}}^{(1)}\}_{jj} + (1 - \delta_{ij}) \, \{W_{\text{off-d}}^{(1)}\}_{ij}, \tag{2.47}$$

$$\{W_{\text{diag}}^{(1)}\}_{jj} = \frac{2}{\beta} (\check{\Theta}_j^2 \, \hat{G}''_{jj}(x) - H''_{jj}(0)), \tag{2.48}$$

$$\{W_{\text{off-d}}^{(1)}\}_{ij} := \frac{2}{\beta} (\Theta_i \Theta_j \, H''_{ij}(x) - H''_{ij}(0)). \tag{2.49}$$

2.6. Form Bounds on the Perturbation

The main technical result in ref. 4 is Theorem 2.1 which we quote in a special case ($\kappa = 0$, see ref. 4, Eq. (II.2)).

Theorem 2.8. Assume Hypotheses 1, 2, and 3. There exist universal constants $\alpha_0, \beta_0, C > 0$ such that, for all $0 \leqslant \alpha \leqslant \alpha_0$ and all $\beta \geqslant \beta_0$, we have

$$\pm W^{(1)}_{H,Q} \leqslant \frac{C}{\beta^{1/2}} A^{(1)}_{H,Q}, \tag{2.50}$$

in the sense of quadratic forms.

Note that, as a consequence, for all $0 \leqslant \alpha \leqslant \alpha_0$ and all $\beta \geqslant \beta_0$, we have

$$\Delta^{(1)}_{H,Q} \geqslant \frac{2(1-\beta^{-\frac{1}{2}}C)\, \lambda_{\min}}{\beta} \mathbb{1}, \tag{2.51}$$

where the lowest eigenvalue $\lambda_{\min} := \inf \sigma(H''(0)) \geqslant C_f/2 > 0$ (see first paragraph of Sec. 3) of the Hessian of H at $x = 0$ is strictly positive, for small α.

It turns out that Theorem 2.8 is not precise enough for the derivation of the correlation asymptotics. What we really need is the following bound whose proof, sketched below, is based on the constructions in ref. 4.

Theorem 2.9. Assume Hypotheses 1, 2, and 3. There exist universal constants $\alpha_0, \beta_0, C > 0$ such that, for all $0 \leqslant \alpha \leqslant \alpha_0$, $\beta > \beta_0$, and all $i, j \in \Lambda$, we have

$$\|(\tilde{A}_i + \beta^{-1})^{-1/2} \{W^{(1)}_{H,Q}\}_{ij} (\tilde{A}_j + \beta^{-1})^{-1/2}\|_{L^2(\mathbb{R}^\Lambda)} \leqslant \frac{C}{\beta^{1/2}} (\delta_{ij} + \alpha a_{ij}). \tag{2.52}$$

For the derivation of Theorem 2.9, we use the matrix $\underline{s} := (s_{ij})_{i,j \in \Lambda}$, defined in (1.11), and the functions

$$J_j[\underline{s}] := \check{\Theta}^2_j \, |\hat{g}'_j| + \sum_{k(\neq j)} s_{jk} \, \Theta^2_k \, |g'_k|. \tag{2.53}$$

The result of the estimates in ref. 4, Eqs. (II.14)–(II.27) can be rephrased as follows,

Lemma 2.10. Assume Hypotheses 1, 2, and 3. There exist universal constants $\alpha_0, \beta_0, C > 0$ such that, for all $0 \leqslant \alpha \leqslant \alpha_0$, all $\beta \geqslant \beta_0$, and all $j \in \Lambda$, we have

$$J_j[\underline{s}] \leqslant \frac{C}{\beta^{1/2}} (\tilde{A}_j + \beta^{-1}), \tag{2.54}$$

in the sense of quadratic forms.

Next, we recall the statement of ref. 4, Lemma II.5.

Lemma 2.11. Assume Hypotheses 1, 2, and 3. For some universal constant C, we have

$$|\check{\Theta}_j^2(x_j)\,\hat{G}''_{jj}(x)-H''_{jj}(0)|\leqslant CJ_j[\underline{s}](x), \tag{2.55}$$

$$|\Theta_j^2(x_j)\,G''_{jj}(x)-H''_{jj}(0)|\leqslant CJ_j[\underline{s}](x). \tag{2.56}$$

Proof. In ref. 4, Lemma II.5, the left sides of (2.55) and (2.56) are bounded by $C'J_j[\underline{a}](x)$. To bound this quantity by $CJ_j[\underline{s}](x)$, we additionally observe that $a_{ij}\leqslant C_a s_{ij}$, which implies $J_j[\underline{a}](x)\leqslant(1+C_a)\,J_j[\underline{s}](x)$, for all $x\in\mathbb{R}^{\Lambda}$. ∎

Note that Lemma 2.11 implies that

$$\|J_j[\underline{s}]^{-1/2}\,\{W^{(1)}_{\text{diag}}\}_{jj}\,J_j[\underline{s}]^{-1/2}\|\leqslant\frac{C}{\beta^{1/2}}, \tag{2.57}$$

for some universal $C<\infty$ and all $j\in\Lambda$. Next, we consider $W^{(1)}_{\text{off-d}}$.

Lemma 2.12. Assume Hypotheses 1, 2, and 3. For some universal constant C and all $i, j\in\Lambda$, we have

$$\|J_i[\underline{s}]^{-1/2}\,\{W^{(1)}_{\text{off-d}}\}_{ij}\,J_j[\underline{s}]^{-1/2}\|\leqslant\frac{C\alpha}{\beta^{1/2}}\,a_{ij}. \tag{2.58}$$

Proof. As in ref. 4, Eq. (II.36), we have

$$|\Theta_i\Theta_j\,\partial^2_{12}w_{ij}(x)-\partial^2_{12}w_{ij}(0)|\leqslant Ca_{ij}(\Theta_i\,|g'_i|+\Theta_j\,|g'_j|+\Theta_i\Theta_j\,|g'_i|^{1/2}\,|g'_j|^{1/2}), \tag{2.59}$$

for all $i\neq j$. Now, we use $a_{ij}=a_{ij}s_{ij}^{1/2}=a_{ij}s_{ij}$, $s_{ij}=s_{ji}$, and $|g'_j|\leqslant c_g^{-1}\,|\hat{g}'_j|$ (cf. ref. 4, Eq. (1.30)), which yield

$$a_{ij}\Theta_i\,|g'_i|=a_{ij}\,|g'_i|^{1/2}\,(s_{ji}\Theta_i^2\,|g'_i|)^{1/2}\leqslant\frac{a_{ij}}{c_g^{1/2}}\,J_i[\underline{s}]^{1/2}\,J_j[\underline{s}]^{1/2}, \tag{2.60}$$

$$a_{ij}\Theta_j|g'_j|=a_{ij}\,(s_{ij}\Theta_j^2\,|g'_j|)^{1/2}\,|g'_j|^{1/2}\leqslant\frac{a_{ij}}{c_g^{1/2}}\,J_i[\underline{s}]^{1/2}\,J_j[\underline{s}]^{1/2}, \tag{2.61}$$

and

$$\begin{aligned} a_{ij}\Theta_i\Theta_j\,|g'_i|^{1/2}\,|g'_j|^{1/2}&=a_{ij}(s_{ij}\Theta_j^2\,|g'_j|)^{1/2}\,(s_{ji}\Theta_i^2\,|g'_i|)^{1/2}\\ &\leqslant c_g^{-\frac{1}{2}}\,a_{ij}J_i[\underline{s}]^{1/2}\,J_j[\underline{s}]^{1/2}. \quad ∎ \end{aligned} \tag{2.62}$$

Proof of Theorem 2.9. The asserted estimate (2.52) follows directly from combining Eq. (2.57), Lemma 2.12, and Lemma 2.10. ∎

3. GREEN'S FUNCTION ESTIMATES

In this section, we study the stability of the Green's function $(\mathbb{1}-\alpha T)^{-1}$ under two types of perturbations: $T \to e^g T$ and $T \to T+gY$, where g is a small parameter, Y is a matrix with no definite sign, and T is the real symmetric $\Lambda \times \Lambda$ transition matrix defined by

$$T_{ij} := \frac{-\partial_{12}^2 w_{ij}(0,0)}{\sqrt{H''_{ii}(0)\, H''_{jj}(0)}} \geqslant 0, \tag{3.1}$$

for all $i, j \in \Lambda$. To justify (3.1), we note that due to Hypothesis 1, 2, and 3, $c_f - \alpha C_w C_a \leqslant H''_{ii}(0) \leqslant C_f + \alpha C_w C_a$ for all $i \in \Lambda$, and hence $0 < c_f/2 \leqslant H''_{ii}(0) \leqslant 2C_f$, provided $0 \leqslant \alpha \leqslant c_f/(2C_w C_a)$. Since $0 \leqslant \max_i \sum_{j(\neq i)} |H''_{ij}(0)| \leqslant \alpha C_a$, we further have the quadratic form bound

$$\frac{C_f}{2} \leqslant C_f - \alpha(C_w+1)C_a \leqslant H''(0) \leqslant C_f + \alpha(C_w+1)C_a \leqslant 2C_f, \tag{3.1a}$$

provided $0 \leqslant \alpha \leqslant C_f/(2(C_w+1)C_a)$, which implies that

$$\frac{C_f}{2} \leqslant \lambda_{\min} := \inf \sigma(H''(0)) \leqslant 2C_f \tag{3.1b}$$

Observe also that $T_{ii} = 0$, for all $i \in \Lambda$. The formula

$$H''_{ij}(0) = H''_{ii}(0)^{1/2} \{\mathbb{1} - \alpha T\}_{ij}\, H''_{jj}(0)^{1/2} \tag{3.2}$$

relates the Green's function $(\mathbb{1}-\alpha T)^{-1}$ to $H''(0)^{-1}$, which defines the metric ρ_H, cf. (1.17). We denote $C_t := \max_i \sum_j T_{ij}$ and observe that $C_t \leqslant 4C_a/C_f$, provided $0 \leqslant \alpha C_f/(2C_w C_a)$.

3.1. Stability under Perturbations of the Form $T \to T + gY$

Our first goal is the derivation of estimates on the matrix elements of resolvent of the form

$$R[\alpha T, g, \vartheta] := (\mathbb{1} - \alpha T - g(\mathbb{1} - \vartheta\alpha T)^{-1})^{-1}, \tag{3.3}$$

where $0 \leqslant \vartheta < 1$. Note that the off-diagonal matrix elements of $(\mathbb{1} - \vartheta\alpha T)^{-1}$ decay faster than those of $(\mathbb{1} - \alpha T)^{-1}$. Hence we expect the decay of $R[\alpha T, g, \vartheta]$ to be dominated by the decay of $(\mathbb{1} - \alpha T)^{-1}$. We quantify this by deriving nontrivial upper and lower bounds on the matrix elements of $R[\alpha T, g, \vartheta]$.

Lemma 3.1. Assume Hypotheses 1, 2, 3, and that $0 \leqslant \alpha C_t \leqslant 1/4$. Let $0 \leqslant \vartheta < 1$, and set $\kappa(\vartheta) := 6(1-\vartheta)^{-2}$. Then, for all $|g| \leqslant (1-\vartheta)^2/6$,

$$(1-\kappa(\vartheta)\,|g|)\left\{\left(\mathbb{1}-\frac{\alpha}{1+2(1-\vartheta)^{-1}\,|g|}\,T\right)^{-1}\right\}_{ij}$$

$$\leqslant R[\alpha T, g, \vartheta]_{ij} \leqslant (1+\kappa(\vartheta)\,|g|)\left\{\left(\mathbb{1}-\frac{\alpha}{1-2(1-\vartheta)^{-1}\,|g|}\,T\right)^{-1}\right\}_{ij}, \quad (3.4)$$

for all $i, j \in \Lambda$.

Remark. The proof we give yields the lemma with the factor 6 replaced by 20. The reader can check the stated bound by replacing the estimate $1-|t| \leqslant \sqrt{1+t} \leqslant 1+|t|$, for $|t| \leqslant 1$, which is used to derive (3.7), by the stronger but asymmetric estimate $1-\frac{2}{3}\,|t| \leqslant \sqrt{1+t} \leqslant 1+\frac{1}{2}\,|t|$, for $|t| \leqslant \frac{3}{4}$.

Proof. For the proof of (3.4), it is convenient to replace αT by a complex variable and consider the complex rational function $f(z) := R[z, g, \vartheta]$ given by

$$f(z) := \frac{1}{1-z-g(1-\vartheta z)^{-1}} = \frac{1-\vartheta z}{\vartheta}\left(z^2 - \frac{1+\vartheta}{\vartheta}\,z + \frac{1-g}{\vartheta}\right)^{-1}$$

$$= \left(\frac{1-\vartheta z}{\vartheta}\right)\left(\frac{1}{\zeta_+ - z}\right)\left(\frac{1}{\zeta_- - z}\right), \quad (3.5)$$

where

$$\zeta_\pm = \frac{1}{2\vartheta}\left[1+\vartheta \pm (1-\vartheta)\sqrt{1+4\vartheta(1-\vartheta)^{-2}\,g}\,\right]. \quad (3.6)$$

Note that the condition $|g| \leqslant (1-\vartheta)^2/20$ insures that $\zeta_\pm \in \mathbb{R}$ and that

$$1-4\vartheta(1-\vartheta)^{-2}\,|g| \leqslant \sqrt{1+4\vartheta(1-\vartheta)^{-2}\,g} \leqslant 1+4\vartheta(1-\vartheta)^{-2}\,|g|, \quad (3.7)$$

which together with $4\vartheta(1-\vartheta) \leqslant 1$ implies the following bounds,

$$\frac{1}{2\vartheta} \leqslant \frac{1}{\vartheta} - \frac{2\,|g|}{1-\vartheta} \leqslant \zeta_+ \leqslant \frac{1}{\vartheta} + \frac{2\,|g|}{1-\vartheta} \leqslant \frac{3}{2\vartheta}, \quad (3.8)$$

$$\frac{1}{2} \leqslant 1 - \frac{2\,|g|}{1-\vartheta} \leqslant \zeta_- \leqslant 1 + \frac{2\,|g|}{1-\vartheta} \leqslant \frac{3}{2}, \quad (3.9)$$

$$0 < \frac{1-\vartheta}{\vartheta}\left(1 - \frac{4\vartheta\,|g|}{(1-\vartheta)^2}\right) \leqslant \zeta_+ - \zeta_- \leqslant \frac{1-\vartheta}{\vartheta}\left(1 + \frac{4\vartheta\,|g|}{(1-\vartheta)^2}\right). \quad (3.10)$$

After some algebra, we arrive at

$$(\zeta_+ - \zeta_-)\,f(z) = \frac{1-\vartheta\zeta_-}{\vartheta\zeta_-}\left(1-\frac{z}{\zeta_-}\right)^{-1} - \frac{1-\vartheta\zeta_+}{\vartheta\zeta_+}\left(1-\frac{z}{\zeta_+}\right)^{-1}. \quad (3.11)$$

Note that (3.11) yields an identity for the matrix $R[\alpha T, g, \vartheta]$ by substituting the matrix αT for z on its right side, invoking functional calculus, or comparing norm-convergent power series. Recall $0 \leqslant \zeta_+^{-1}\alpha T < \zeta_-^{-1}\alpha T \leqslant \frac{1}{2}\mathbb{1}$, cf. (3.10) and (3.8). We now bound the matrix elements of the resolvents $(1-\zeta_\pm^{-1}\alpha T)^{-1}$. Since $\zeta_+ > \zeta_-$ we have, by comparing Neumann series,

$$0 \leqslant \{(1-\zeta_+^{-1}\,\alpha T)^{-1}\}_{ij} \leqslant \{(1-\zeta_-^{-1}\,\alpha T)^{-1}\}_{ij}. \tag{3.12}$$

In order to bound matrix elements of $f(\alpha T)$ we use the following two bounds

$$\frac{1-\vartheta}{\vartheta}\left(1-\frac{4\,|g|}{(1-\vartheta)^2}\right) \leqslant \frac{1-\vartheta\zeta_-}{\vartheta\zeta_-} \leqslant \frac{1-\vartheta}{\vartheta}\left(1+\frac{4\,|g|}{(1-\vartheta)^2}\right), \tag{3.13}$$

$$-\frac{1-\vartheta}{\vartheta}\cdot\frac{4\,|g|}{(1-\vartheta)^2} \leqslant \frac{1-\vartheta\zeta_+}{\theta\zeta_+} \leqslant \frac{1-\vartheta}{\vartheta}\cdot\frac{4\,|g|}{(1-\vartheta)^2}. \tag{3.14}$$

The reader may readily derive these two bounds using (3.8) and (3.9) together with the following simple observation: $t \to (1-t)/t$ is monotonically decreasing, and for $|t| \leqslant 1/2$ we have $1-2\,|t| \leqslant (1+t)^{-1} \leqslant 1+2\,|t|$.

From (3.11)–(3.14) we now deduce that

$$\begin{aligned}(\zeta_+-\zeta_-)^{-1}\,\frac{1-\vartheta}{\vartheta}\left(1-\frac{8\,|g|}{(1-\vartheta)^2}\right)\{(1-\zeta_-^{-1}\,\alpha T)^{-1}\}_{ij}\\ \leqslant R[\alpha T, g, \vartheta]_{ij} \leqslant (\zeta_+-\zeta_-)^{-1}\,\frac{1-\vartheta}{\vartheta}\left(1+\frac{8\,|g|}{(1-\vartheta)^2}\right)\{(1-\zeta_-^{-1}\,\alpha T)^{-1}\}_{ij},\end{aligned} \tag{3.15}$$

which, together with (3.10), yields the claim. ∎

Lemma 3.1 is an important input for the proof of the following lemma, because it insures the positivity of the matrix elements of $R[\alpha T, g, \vartheta]$.

Lemma 3.2. Assume Hypotheses 1, 2, 3, and that $0 \leqslant \alpha C_t \leqslant 1/4$. Let $0 \leqslant \vartheta < 1$, set $\kappa(\vartheta) := 12(1-\vartheta)^{-3}$, and assume $|g| \leqslant (1-\vartheta)^3/6$. Suppose that Y is a real $\Lambda\times\Lambda$ matrix obeying

$$\forall i, j \in \Lambda\colon\quad |Y_{ij}| \leqslant \{(\mathbb{1}-\vartheta\alpha T)^{-1}\}_{ij}. \tag{3.16}$$

Then $\mathbb{1}-\alpha T - gY$ is invertible, and the matrix elements of its resolvent fulfill the following estimates.

$$R[\alpha T, -|g|, \vartheta]_{ij} \leqslant \{(\mathbb{1}-\alpha T-gY)^{-1}\}_{ij} \leqslant R[\alpha T, |g|, \vartheta]_{ij}, \tag{3.17}$$

for all $i, j \in \Lambda$.

Proof. First, we expand the inverse of $\mathbb{1}-\alpha T-gY$ in a Neumann series and use the upper bound on Y_{ij} in (3.16) to obtain the upper bound asserted in (3.17),

$$\begin{aligned}\{(\mathbb{1}-\alpha T-gY)^{-1}\}_{ij} &= \sum_{n=0}^{\infty} \{(\mathbb{1}-\alpha T)^{-1}\,[gY\,(\mathbb{1}-\alpha T)^{-1}]^{n}\}_{ij} \\ &\leqslant \sum_{n=0}^{\infty} \{(\mathbb{1}-\alpha T)^{-1}\,[|g|\,(\mathbb{1}-\vartheta\alpha T)^{-1}\,(\mathbb{1}-\alpha T)^{-1}]^{n}\}_{ij} \\ &= R[\alpha T,|g|,\vartheta]_{ij}. \end{aligned} \tag{3.18}$$

The lower bound in (3.17) follows similarly from a Neumann series,

$$\begin{aligned}&\{(\mathbb{1}-\alpha T-gY)^{-1}\}_{ij} \\ &\quad= \sum_{n=0}^{\infty} \{R[\alpha T,-|g|,\vartheta]\,[|g|\,((\mathbb{1}-\vartheta\alpha T)^{-1}-Y)\,R[\alpha T,-|g|,\vartheta]]^{n}\}_{ij} \\ &\quad\geqslant R[\alpha T,-|g|,\vartheta]_{ij}, \end{aligned} \tag{3.19}$$

retaining from the series only the term corresponding to $n=0$. Here we use the positivity of the matrix elements of $(\mathbb{1}-\vartheta\alpha T)^{-1}-Y$, following from (3.16), as well as the positivity of the matrix elements of $R[\alpha T,-|g|,\vartheta]$, which follows from Lemma 3.1. ∎

Putting together Lemma 3.2 and Lemma 3.1, we arrive at the first main result of this subsection.

Theorem 3.3. Assume Hypotheses 1, 2, 3, and that $0\leqslant\alpha C_t\leqslant 1/4$. Let $0\leqslant\vartheta<1$, set $\kappa(\vartheta):=12(1-\vartheta)^{-3}$, and assume $|g|\leqslant(1-\vartheta)^3/6$. Suppose that Y is a real $\Lambda\times\Lambda$ matrix obeying

$$|Y_{ij}|\leqslant\{(\mathbb{1}-\vartheta\alpha T)^{-1}\}_{ij}. \tag{3.20}$$

Then $\mathbb{1}-\alpha T-gY$ is invertible, and the matrix elements of its resolvent fulfill the following estimates.

$$\begin{aligned}&(1-\kappa(\vartheta)\,|g|)\left\{\left(\mathbb{1}-\frac{\alpha T}{1+2(1-\vartheta)^{-1}\,|g|}\right)^{-1}\right\}_{ij} \\ &\quad\leqslant\{(\mathbb{1}-\alpha T-gY)^{-1}\}_{ij}\leqslant(1+\kappa(\vartheta)\,|g|)\left\{\left(\mathbb{1}-\frac{\alpha T}{1-2(1-\vartheta)^{-1}|g|}\right)^{-1}\right\}_{ij}, \end{aligned} \tag{3.21}$$

for all $i,j\in\Lambda$.

For the derivation of the correlation asymptotics we actually need a more refined version of the Neumann series expansion in (3.19) which yields the following estimate.

Theorem 3.4. Assume Hypotheses 1, 2, 3, and that $0 \leqslant \alpha C_t \leqslant 1/4$. Let $0 \leqslant \vartheta < 1$, set $\kappa(\vartheta) := 12(1-\vartheta)^{-3}$, and assume $|g| \leqslant (1-\vartheta)^3/12$. Suppose that S and Y are real $\Lambda \times \Lambda$ matrices obeying

$$|S_{ij}|, |Y_{ij}| \leqslant \{(\mathbb{1} - \vartheta\alpha T)^{-1}\}_{ij}. \tag{3.22}$$

Then we have the following estimate,

$$|\{S\,(\mathbb{1} - \alpha T - gY)^{-1}\}_{ij}| \leqslant |g|^{-1}\,\{(\mathbb{1} - \alpha T - gY)^{-1}\}_{ij}, \tag{3.23}$$

for all $i, j \in \Lambda$.

Proof. Clearly, by replacing Y by $-Y$, we may assume without loss of generality that $g \geqslant 0$. Due to Assumption (3.22), we have that

$$|S_{ij}| \leqslant X_{ij} := \{2(\mathbb{1} - \vartheta\alpha T)^{-1} - Y\}_{ij}. \tag{3.24}$$

Moreover, $(\mathbb{1} - \alpha T - gY)^{-1}$ has nonnegative matrix elements, by Theorem 3.3 and Lemma 3.2. The latter also implies that $R[\alpha T, -2g, \vartheta]_{ij} \geqslant \delta_{ij}$. Therefore,

$$\begin{aligned}
|\{S\,(\mathbb{1} - \alpha T - gY)^{-1}\}_{ij}| &\leqslant \{X\,(\mathbb{1} - \alpha T - gY)^{-1}\}_{ij} \\
&= \{X(\mathbb{1} - \alpha T + 2g(1 - \vartheta\alpha T)^{-1} - gX)^{-1}\}_{ij} \\
&= \frac{1}{g}\left\{\sum_{n=1}^{\infty} (gX\, R[\alpha T, -2g, \vartheta])^n\right\}_{ij} \\
&\leqslant \frac{1}{g}\left\{\sum_{n=0}^{\infty} R[\alpha T, -2g, \vartheta](gX R[\alpha T, -2g, \vartheta])^n\right\}_{ij} \\
&= \frac{1}{g}\{(\mathbb{1} - \alpha T - gY)^{-1}\}_{ij},
\end{aligned} \tag{3.25}$$

proving the claim. ∎

3.2. Stability under Perturbations of the Form $T \to e^g T$

In this subsection we study stability of the correlations under scaling of the transition matrix T defined in (3.1). We begin with a log-convexity estimate, which is similar to ref. 3, (VI.42).

Lemma 3.5. Suppose Hypothesis 2 and that $\alpha C_t \leqslant 1/4$. Then there exists a universal constant $g_0 > 0$ such that, for $g_1, g_2 \in [-g_0, g_0]$, $0 \leqslant \kappa \leqslant 1$, and all $i, j \in \Lambda$, we have

$$\{(\mathbb{1} - e^{\kappa g_1 + (1-\kappa) g_2} \alpha T)^{-1}\}_{ij} \leqslant (\{(\mathbb{1} - e^{g_1} \alpha T)^{-1}\}_{ij})^{\kappa} (\{(\mathbb{1} - e^{g_2} \alpha T)^{-1}\}_{ij})^{1-\kappa}. \tag{3.26}$$

Proof. Note that all the resolvents can be expanded in norm-convergent Neumann series thanks to the condition $\alpha C_t \leqslant 1/4$. Then we obtain

$$\begin{aligned}\{(\mathbb{1} - e^{\kappa g_1 + (1-\kappa) g_2} \alpha T)^{-1}\}_{ij} &= \sum_{n=0}^{\infty} (e^{g_1 n}\{T^n\}_{ij})^{\kappa} (e^{g_2 n} \{T^n\}_{ij})^{1-\kappa} \\ &\leqslant \left(\sum_{n=0}^{\infty} e^{g_1 n} \{T^n\}_{ij}\right)^{\kappa} \left(\sum_{n=0}^{\infty} e^{g_2 n} \{T^n\}_{ij}\right)^{1-\kappa},\end{aligned} \tag{3.27}$$

using Hölder's inequality. ∎

We also use the following elementary result

Lemma 3.6. Let $f\colon [0, \infty) \to \mathbb{R}$ with $f(0) = 0$. If f is convex then

$$\forall a, b \geqslant 0\colon \quad f(a+b) \geqslant f(a) + f(b), \tag{3.28}$$

and if f is concave then

$$\forall a, b \geqslant 0\colon \quad f(a+b) \leqslant f(a) + f(b). \tag{3.29}$$

Proof. We consider only the convex case. The concave case follows from replacing f by $-f$. We can assume without loss of generality that $0 \leqslant a \leqslant b$ and that $b > 0$. Writing

$$a = \frac{a}{b} \cdot b + \frac{b-a}{b} \cdot 0 \qquad \text{and} \qquad b = \frac{b}{a+b} \cdot (a+b) + \frac{a}{a+b} \cdot 0, \tag{3.30}$$

the convexity of f implies that

$$f(a) + f(b) \leqslant \left(1 + \frac{a}{b}\right) f(b) \leqslant \left(1 + \frac{a}{b}\right) \frac{b}{a+b} f(a+b) = f(a+b). \tag{3.31}$$

This completes the proof. ∎

We now turn to the main result of this subsection

Theorem 3.7. Suppose Hypothesis 2 and $\alpha C_t \leqslant 1/4$. Then there exist universal constants $g_0 > 0$ and C, such that, for $0 \leqslant g \leqslant g_0$, we have

$$\{(\mathbb{1} - e^g \alpha T)^{-1}\}_{ij} \leqslant (1 + Cg)\, \{(\mathbb{1} - \alpha T)^{-1}\}_{ij}^{1-Cg}, \tag{3.32}$$

$$(1 - Cg)\, \{(\mathbb{1} - \alpha T)^{-1}\}_{ij}^{1+Cg} \leqslant \{(\mathbb{1} - e^{-g} \alpha T)^{-1}\}_{ij}. \tag{3.33}$$

Assume additionally that either $T_{ij} = 0$ or $T_{ij} \geqslant \sigma_T$, for some universal constant $\sigma_T > 0$. Then there exists a universal constant $g_0 > 0$, such that, for $0 \leqslant g \leqslant g_0$:

$$\{(\mathbb{1} - \alpha T)^{-1}\}_{ij}^{1-u(\alpha\sigma_T)\, g} \leqslant \{(\mathbb{1} - e^g \alpha T)^{-1}\}_{ij}, \tag{3.34}$$

$$\{(\mathbb{1} - e^{-g} \alpha T)^{-1}\}_{ij} \leqslant \{(\mathbb{1} - \alpha T)^{-1}\}_{ij}^{1+u(\alpha\sigma_T)\, g}, \tag{3.35}$$

where $u(\alpha\sigma_T) := 1/\ln[1/(\alpha\sigma_T)] > 1/2$.

Remark. In applications we will use this theorem with e^g replaced by $1+g$ and e^{-g} replaced by $1-g$. The corresponding estimates are clearly equivalent (but with different g_0's). Also notice that $\alpha\sigma_T \leqslant \alpha C_t \leqslant 1/4$.

Proof. An application of Lemma 3.5 with $\kappa := g/g_0$, $g_1 := g_0$ (so that $\kappa g_1 = g$), and $g_2 = 0$ yields

$$\begin{aligned} \{(\mathbb{1} - e^g \alpha T)^{-1}\}_{ij} &\leqslant (\{(\mathbb{1} - e^{g_0} \alpha T)^{-1}\}_{ij})^{g/g_0} (\{(\mathbb{1} - \alpha T)^{-1}\}_{ij})^{1-g/g_0} \\ &\leqslant (1 - e^{g_0} \alpha C_a)^{-g/g_0} (\{(\mathbb{1} - \alpha T)^{-1}\}_{ij})^{1-g/g_0}, \end{aligned} \tag{3.36}$$

and hence (3.32). Equation (3.36) also implies (3.33) upon the substitution $\alpha \to \alpha' := e^g \alpha$.

To prove (3.35), we note that $\alpha\sigma_T \leqslant \alpha \max_{i,j} T_{i,j} \leqslant \alpha C_t \leqslant 1/4$. Hence $u(\alpha\sigma_T) = 1/\ln[1/(\alpha\sigma_T)] \geqslant 1/\ln[4] > 1/2$. We consider $b = (i, j) \in \mathscr{B}_a$, i.e., $T_{ij} \neq 0$. Then, by assumption, $\alpha T_{ij} \geqslant \alpha\sigma_T$, and we conclude that

$$e^{-g} \alpha T_{ij} \leqslant (\alpha T_{ij})^{u(\alpha\sigma_T)\, g}. \tag{3.37}$$

Next we (repeatedly) use Lemma 3.6 with the function $t \mapsto t^{1+u(\alpha\sigma_T)\, g}$, which is convex and vanishes at 0 and where $u(\alpha\sigma_T) > 0$. An expansion of the resolvent matrix elements in terms of paths thus gives the desired estimate,

$$\begin{aligned} &\{(\mathbb{1} - e^{-g} \alpha T)^{-1}\}_{ij} \\ &= \delta_{ij} + \sum_{\gamma \in \Gamma(i,j)} \prod_{b \in \gamma} e^{-g} \alpha T_b \leqslant \delta_{ij} + \sum_{\gamma \in \Gamma(i,j)} \left(\prod_{b \in \gamma} \alpha T_b \right)^{1+u(\alpha\sigma_T)\, g} \\ &\leqslant \left(\delta_{ij} + \sum_{\gamma \in \Gamma(i,j)} \prod_{b \in \gamma} \alpha T_b \right)^{1+u(\alpha\sigma_T)\, g} \\ &= (\{(\mathbb{1} - \alpha T)^{-1}\}_{ij})^{1+u(\alpha\sigma_T)\, g}, \end{aligned} \tag{3.37a}$$

additionally taking into account that $i \neq j$, since $T_{ij} \neq 0$. The remaining inequality (3.34) is proved analogously, using the concave function $t \mapsto t^{1-u(\alpha\sigma_T)\, g}$. ∎

3.3. Green's Functions and Associated Metrics

In this section, we study resolvents $(\mathbb{1} - \alpha T)^{-1}$, where the transition matrix T is defined in (3.1), and $\alpha \geqslant 0$ is sufficiently small such that $\alpha C_a \leqslant 1/4$. Since $(\mathbb{1} - \alpha T)^{-1}$ has positive matrix elements, the expression

$$\exp[-\rho_H(i, j)] := \frac{\{(\mathbb{1} - \alpha T)^{-1}\}_{ij}}{\{(\mathbb{1} - \alpha T)^{-1}\}_{ii}^{1/2}\ \{(\mathbb{1} - \alpha T)^{-1}\}_{jj}^{1/2}} \tag{3.39}$$

defines a function $\rho_H \colon \Lambda \times \Lambda \to [0, \infty]$. Note that by (3.2), this definition coincides with (1.17). It is a remarkable fact that ρ_H is actually a metric on Λ.

Theorem 3.8. Assume Hypotheses 1, 2, 3, and that $0 \leqslant \alpha C_a \leqslant 1/4$. Then $\rho_H \colon \Lambda \times \Lambda \to [0, \infty]$ is a metric on Λ.

Proof. For the proof, we denote $R := (\mathbb{1} - \alpha T)^{-1}$. Its matrix elements are nonnegative, and R is positive, as a quadratic form. The symmetry $\rho_H(i, j) = \rho_H(j, i)$ is trivial, since T is symmetric.

Since R is positive, as a quadratic form,

$$R_{ij} = \langle R^{1/2} e_i \mid R^{1/2} e_j \rangle \leqslant \|R^{1/2} e_i\|\, \|R^{1/2} e_j\| = R_{ii} R_{jj}, \tag{3.40}$$

with equality iff $R^{1/2} e_i$ and $R^{1/2} e_j$ are parallel, which is equivalent to e_i and e_j being parallel, i.e., $i = j$. Therefore, $\rho_H \geqslant 0$, and $\rho_H(i, j) = 0$ iff $i = j$.

As for the triangle inequality

$$\rho_H(i, j) \leqslant \rho_H(i, k) + \rho_H(k, j), \tag{3.41}$$

which is equivalent to

$$R_{ik}\, R_{kj} \leqslant R_{ij}\, R_{kk}, \tag{3.42}$$

we note that it is sufficient to consider the case where $i, j, k \in \Lambda$ are three different points.

Expanding in a Neumann series, we see that $T_{ij} \geqslant 0$ implies that $R_{kk} \geqslant \{\mathbb{1}\}_{kk} = 1$. Moreover, we may expand R_{ij} as a sum over all paths γ from i to j,

$$R_{ij} = \delta_{ij} + \sum_{\gamma \in \Gamma(i, j)} \prod_{b \in \gamma} T_b. \tag{3.43}$$

We recall that a path γ is a (ordered) nonempty, finite collection of nearest-neighbour bonds of the form $\gamma = \{(i_0, i_1), (i_1, i_2), \ldots, (i_{n-1}, i_n)\} \subseteq \mathscr{B}_a$, with $i_0 = i$ and $i_n = j$. The collection of all paths from i to j is denoted $\Gamma(i, j)$. We further introduce the set $\Gamma'(i, j) \subset \Gamma(i, j)$ of paths from i to j which do not visit j in between. So, if $\gamma = \{(i_0, i_1), (i_1, i_2), \ldots, (i_{n-1}, i_n)\} \in \Gamma'(i, j)$, then $i_0 = i$, $i_n = j$, and $i_1 \neq j, \ldots, i_{n-1} \neq j$.

We define the concatenation $\circ : \Gamma(i, j) \times \Gamma(j, k) \to \Gamma(i, k)$ of two paths in the obvious way, i.e., $\gamma_1 \circ \gamma_2 := (b_1, \ldots, b_{m+n})$, for $\gamma_1 = (b_1, \ldots, b_m) \in \Gamma(i, j)$ and $\gamma_2 = (b_{m+1}, \ldots, b_{m+n}) \in \Gamma(j, k)$. Given two points $i, j \in \Lambda$, we observe the following disjoint decomposition identity,

$$\Gamma(i, j) = \Gamma'(i, j) \cup (\Gamma'(i, j) \circ \Gamma(j, j)). \tag{3.44}$$

Thus, defining a Green's function R' by

$$R'_{ij} := \sum_{\gamma' \in \Gamma'(i, j)} \prod_{b' \in \gamma'} T_{b'}, \tag{3.45}$$

we have the following identity

$$R_{ik} = \delta_{ik} + R'_{ik} + R'_{ik}\,(R_{kk} - 1) = \delta_{ik} + R'_{ik}\,R_{kk}, \tag{3.46}$$

for all $i, k \in \Lambda$. Now suppose that $i, j, k \in \Lambda$ are three different points in the lattice. Then, the concatenation $\circ$, viewed as a map $\circ : \Gamma'(i, k) \times \Gamma(k, j) \to \Gamma(i, j)$ is injective. This implies that $R'_{ik} R_{kj} \leqslant R_{ij}$. Therefore, using (3.46), we observe that

$$R_{ik} R_{kj} = R'_{ik} R_{kk} R_{kj} \leqslant R_{ij} R_{kk}, \tag{3.47}$$

which proves the triangle inequality (3.42). ▮

4. SPECTRAL SEPARATION AND IMPROVED DECAY

4.1. The Feshbach Projection Method

Our analysis of the correlation asymptotics is built upon the Feshbach map associated to the projection

$$P := p \otimes \mathbb{1}, \qquad \text{where} \quad p := \mathscr{Z}_\beta^{-1}\, |e^{-\beta H}\rangle\langle e^{-\beta H}| \tag{4.1}$$

is the rank-1 projection onto $e^{-\beta H} \in L^2(\mathbb{R}^\Lambda)$. We write $\bar{p} := \mathbb{1} - p$ and $\bar{P} := \mathbb{1} - P = \bar{p} \otimes \mathbb{1}$. The Feshbach operator $\mathscr{F}_P(\Delta_{H,Q})$ is defined to be the image of $\Delta_{H,Q}$ under the Feshbach map, see refs. 1–3, and 7 for a detailed description of the Feshbach map and its properties. The Feshbach operator $\mathscr{F}_P := \mathscr{F}_P(\Delta_{H,Q})$: Ran $P \to$ Ran P is now defined as

$$\mathscr{F}_P = P\,\Delta^{(1)}_{H,Q}\,P - \Delta^*_{\bar{P}P}\,(\bar{\Delta}^{(1)}_{H,Q})^{-1}\,\Delta_{\bar{P}P}, \tag{4.2}$$

where we write $\bar{\Delta}^{(1)}_{H,Q} := \bar{P}\Delta^{(1)}_{H,Q}\bar{P}$ and use the overlap operator $\Delta_{\bar{P}P}$: Ran $P \to$ Ran $\bar{P}$ defined by

$$\Delta_{\bar{P}P} := \bar{P}\,\Delta^{(1)}_{H,Q}P. \tag{4.3}$$

Note that $\bar{\Delta}^{(1)}_{H,Q} \geqslant 2\lambda_{\min}(1 - C\beta^{-1/2})\,\beta^{-1}\mathbb{1} > 0$ (cf. Theorem 2.8) is bounded invertible on Ran $\bar{P}$ and that $\Delta_{\bar{P}P}$ is bounded. Hence $\mathscr{F}_P$ is well-defined. One of the crucial properties of the Feshbach map is its isospectrality. That is, $\mathscr{F}_P$ is invertible on Ran P if and only if $\Delta^{(1)}_{H,Q}$ is invertible on $\mathscr{H}^{(1)}$. In this case, we have

$$\begin{aligned}(\Delta^{(1)}_{H,Q})^{-1} &= (P - \bar{P}(\bar{\Delta}^{(1)}_{H,Q})^{-1}\,\Delta_{\bar{P}P})\,\mathscr{F}_P^{-1}\,(P - \Delta^*_{\bar{P}P}(\bar{\Delta}^{(1)}_{H,Q})^{-1}\,\bar{P})\\ &\quad + \bar{P}(\bar{\Delta}^{(1)}_{H,Q})^{-1}\,\bar{P}.\end{aligned} \tag{4.4}$$

The rest of this subsection is devoted to properties of the projection P, in relation to $A_{H,Q}$. We recall that, for $j \in \Lambda$,

$$\tilde{A}_j = \check{A}_j + \sum_{k(\neq j)} A_k, \tag{4.5}$$

$$\check{A}_j = \Theta_j Z_j(H)\,Z_j^*(H)\,\Theta_j - \frac{2}{\beta}\,\check{\Theta}_j^2\hat{G}''_{jj}(x), \tag{4.6}$$

$$A_j = \Theta_j Z_j^*(G)\,Z_j(G)\,\Theta_j. \tag{4.7}$$

On Ran p, we have the following upper bound,

Lemma 4.1. Assume Hypotheses 1, 2, and 3. There exist universal constants $\alpha_0, \beta_0, C > 0$ such that, for all $j \in \Lambda$, $0 \leqslant \alpha < \alpha_0$, and $\beta > \beta_0$, we have

$$\|p\tilde{A}_j p\| \leqslant \frac{C}{\beta^{3/2}}. \tag{4.8}$$

Remark. By virtue of Theorem 2.3, we could have chosen p to be j-dependent projections onto vectors of the form $e^{-\beta(H+r_j)}$, for a large class of r_j's. The choice (4.1) seems the most convenient here. We note that the most desirable choice, $r_j = -q_j$, may cause Lemma 4.1 to be false. The reason is that the expression $p\check{A}_j p$ would contain (the square of) a term of the form $e^{-\beta(H-2q_j)}$, and the Hamiltonian $H - 2q_j$ does not in general localize at 0.

Proof. We first remark that, for $j \in \Lambda$, we have

$$\left(\sum_{k(\neq j)} A_k \right) e^{-\beta H} = \sum_{k(\neq j)} \Theta_k \, Z_k^*(G) \, Z_k(G) \, \Theta_k \, e^{-\beta H} = 0, \tag{4.9}$$

due to (2.36), (2.37), and (2.45). Next, a short computation yields

$$\begin{aligned} \check{A}_j &= \Theta_j Z_j(H) \, Z_j^*(H) \, \Theta_j - \frac{2}{\beta} \check{\Theta}_j^2 \hat{G}_{jj}'' \\ &= \Theta_j Z_j^*(H) \, Z_j(H) \, \Theta_j + \frac{2}{\beta} (\Theta_j^2 G_{jj}'' - \check{\Theta}_j^2 \hat{G}_{jj}'') + \frac{2}{\beta} \Theta_j^2 q_j''. \end{aligned} \tag{4.10}$$

By Lemmata 2.10 and 2.11, we have

$$\left\| \frac{2}{\beta} (\Theta_j^2 G_{jj}'' - \check{\Theta}_j^2 \hat{G}_{jj}'') \right\| \leqslant \frac{C}{\beta^{1/2}} (\tilde{A}_j + \beta^{-1}), \tag{4.11}$$

in the sense of quadratic forms, for some universal C. Inserting (4.11) into (4.10) and sandwiching with $e^{-\beta H}$, we thus obtain, for β sufficiently large,

$$\begin{aligned} \langle e^{-\beta H} \mid \tilde{A}_j e^{-\beta H} \rangle &\leqslant 2 \langle e^{-\beta H} \mid \Theta_j Z_j^*(H) \, Z_j(H) \, \Theta_j e^{-\beta H} \rangle \\ &\quad + \frac{4}{\beta} \|\Theta_j \, |q_j''|^{1/2} \, e^{-\beta H}\|^2 + \frac{C \mathscr{Z}_\beta}{\beta^{3/2}} \\ &= 2 \, \|\Theta_j q_j' \, e^{-\beta H}\|^2 + \frac{4}{\beta} \|\Theta_j \, |q_j''|^{1/2} \, e^{-\beta H}\|^2 + \frac{C \mathscr{Z}_\beta}{\beta^{3/2}} \\ &\leqslant C \left(\frac{\mathscr{Z}_\beta}{\beta^{3/2}} + \int_{|x_j| \geqslant \hat{R}_0} e^{-2\beta(H - q_j)} \, d^\Lambda x \right), \end{aligned} \tag{4.12}$$

using that $\|q_j'\|_\infty, \|q_j''\|_\infty \leqslant C'$ are bounded by some universal constant C' and vanish on $[-\hat{R}_0, \hat{R}_0]$. According to Theorem 2.4, the integral on the right side of (4.12) is bounded by $\mathscr{Z}_\beta \, e^{-\delta\beta}$, for some universal $\delta > 0$. This yields the asserted estimate. ∎

On Ran $\bar{p}$, we have the following complementary lower bound,

Lemma 4.2. Assume Hypotheses 1, 2, and 3. There exist universal constants $\alpha_0, \beta_0, C > 0$ such that, for all $j \in \Lambda$, $0 \leqslant \alpha < \alpha_0$, and $\beta > \beta_0$, we have

$$\bar{p} \tilde{A}_j \bar{p} \geqslant (1 - C\beta^{-\frac{1}{2}}) \, \frac{2\lambda_{\min}}{\beta} \, \bar{p}. \tag{4.13}$$

Proof. We first pick a smooth characteristic function $\chi \in C_0^\infty(\mathbb{R}_0^+; [0,1])$ on the interval $[0, \hat{R}_0)$, such that $\chi \equiv 1$ on $[0, \hat{R}_0/2)$, $\chi \equiv 0$ on $[\hat{R}_0, \infty)$, and $\bar{\chi} := \sqrt{1-\chi^2} \in C^\infty$ is smooth, as well. We denote $\chi_j := \chi(|x_j|)$. The IMS localization formula reads

$$\tilde{A}_j = \chi_j \tilde{A}_j \chi_j + \bar{\chi}_j \tilde{A}_j \bar{\chi}_j - \beta^{-2}((\chi_j')^2 + (\bar{\chi}_j')^2). \tag{4.14}$$

Note that, by Lemma 2.10,

$$\frac{C}{\beta^{1/2}}(\tilde{A}_j + \beta^{-1}) \geqslant J_j[\underline{s}] \geqslant |\hat{g}_j'|, \tag{4.15}$$

for some universal constant $C < \infty$. Furthermore, Eq. (2.9) yields that $|\hat{g}_j'(x_j)| \geqslant c_g \min\{1, |\hat{R}_0|/2\}$, for $|x_j| \geqslant |\hat{R}_0|/2$. Thus, for some universal constant $c > 0$ and β sufficiently large, we have that

$$\bar{\chi}_j \tilde{A}_j \bar{\chi}_j \geqslant \frac{c}{\beta^{1/2}} \bar{\chi}_j^2. \tag{4.16}$$

Next, the supersymmetric property (2.7) implies that

$$\Delta_{H,Q}^{(0)} \geqslant (1 - C\beta^{-\frac{1}{2}}) \frac{2\lambda_{\min}}{\beta} \bar{p}, \tag{4.17}$$

for some universal C. Moreover, $\chi_j \tilde{A}_j \chi_j = \chi_j \, \Delta_{H,Q}^{(0)} \chi_j$, and hence we have the lower bound

$$\chi_j \tilde{A}_j \chi_j \geqslant (1 - C\beta^{-\frac{1}{2}}) \frac{2\lambda_{\min}}{\beta} (\chi_j^2 - \chi_j p \chi_j). \tag{4.18}$$

Putting together (4.14)–(4.18), we have that

$$\begin{aligned} \tilde{A}_j &\geqslant (1 - C\beta^{-\frac{1}{2}}) \frac{2\lambda_{\min}}{\beta} \chi_j^2 + \frac{c}{\beta^{1/2}} \bar{\chi}_j^2 - \frac{2\lambda_{\min}}{\beta} \chi_j p \chi_j - \frac{C}{\beta^2} \\ &\geqslant \frac{2\lambda_{\min}}{\beta} \left\{ 1 - \frac{C'}{\beta^{1/2}} - \chi_j p \chi_j \right\}, \end{aligned} \tag{4.19}$$

for universal $c, C, C' > 0$, and β sufficiently large. Sandwiching Eq. (4.19) with $\bar{p}$, we arrive at

$$\bar{p} \tilde{A}_j \bar{p} \geqslant \frac{2\lambda_{\min}}{\beta} \left\{ 1 - \frac{C'}{\beta^{1/2}} - \|\bar{p} \chi_j p \chi_j \bar{p}\| \right\} \bar{p}. \tag{4.20}$$

Now, observe that due to $\bar{p}p = p\bar{p} = 0$,

$$\bar{p}\chi_j p\chi_j\bar{p} = \bar{p}(1-\chi_j)\, p(1-\chi_j)\,\bar{p}. \tag{4.21}$$

Since $1-\chi_j$ vanishes on $[-\hat{R}_0/2, \hat{R}_0/2]$, there exist a universal $\delta > 0$ such that

$$\begin{aligned}\|\bar{p}\chi_j p\chi_j\bar{p}\| &\leqslant \|(1-\chi_j)\, p(1-\chi_j)\| = \mathscr{Z}_\beta^{-1}\, \|(1-\chi_j)\, e^{-\beta H}\|^2 \\ &\leqslant \int_{|x_j| \geqslant \hat{R}_0/2} e^{-2\beta H}\, \frac{d^\Lambda x}{\mathscr{Z}_\beta} \leqslant e^{-\delta\beta},\end{aligned} \tag{4.22}$$

according to Theorem 2.3. We finally obtain the asserted estimate (4.13) for sufficiently large β by inserting (4.22) into (4.20). ∎

The following Theorem is an immediate consequence of Lemma 4.2 and Theorem 2.8.

Theorem 4.3. Assume Hypotheses 1, 2, and 3. We have the following *spectral separation* estimates: There exist universal constants α_0, β_0, $C > 0$ such that, for any $0 \leqslant \alpha < \alpha_0$ and $\beta > \beta_0$, we have

$$\bar{P}A^{(1)}_{H,Q}\bar{P} \geqslant (1 - C\beta^{-\frac{1}{2}})\, \frac{4\lambda_{\min}}{\beta}\, \bar{P}, \tag{4.23}$$

$$\bar{P}\, \Delta^{(1)}_{H,Q}\bar{P} \geqslant (1 - C\beta^{-\frac{1}{2}})\, \frac{4\,\lambda_{\min}}{\beta}\, \bar{P}. \tag{4.24}$$

4.2. Improved Decay

We begin by introducing some notation. Let

$$D := \sum_j D_j \otimes E_{jj} := D_H^{-\frac{1}{2}} \left(\sum_j \tilde{A}_j \otimes E_{jj} \right) D_H^{-\frac{1}{2}}, \tag{4.25}$$

$$B_0 := D + \frac{2}{\beta}\, \mathbb{1}, \tag{4.26}$$

$$\begin{aligned} B &:= D_H^{-\frac{1}{2}} A^{(1)}_{H,Q} D_H^{-\frac{1}{2}} \\ &= D + \frac{2}{\beta}(\mathbb{1} - \alpha T) = B_0 - \frac{2\alpha}{\beta}\, T, \end{aligned} \tag{4.27}$$

$$V := D_H^{-\frac{1}{2}} W^{(1)}_{H,Q} D_H^{-\frac{1}{2}}, \tag{4.28}$$

where D_H is the diagonal matrix given by

$$D_H := \sum_j H''_{jj}(0) \otimes E_{jj}. \tag{4.29}$$

We frequently use without further comment that

$$(c_f - \alpha C_a)\,\mathbb{1} \leqslant D_H \leqslant (C_f + \alpha C_a)\,\mathbb{1} \tag{4.30}$$

is bounded above and below by universal constants, for small $\alpha > 0$. Furthermore, $D_j := \{D\}_{jj} = H''_{jj}(0)^{-1}\,\tilde{A}_j$, and T is the matrix given by (3.1). We also introduce $\bar{A}^{(1)}_{H,Q} := \bar{P} A^{(1)}_{H,Q} \bar{P}$, $\bar{B} := \bar{P} B \bar{P}$, $\bar{B}_0 := \bar{P} B_0 \bar{P}$, $\bar{D} := \bar{P} D \bar{P}$, $\bar{D}_j := \bar{p} D_j \bar{p}$, $\bar{V} := \bar{P} V \bar{P}$, and $\bar{T} := \bar{p} \otimes T$. We observe that

$$(\bar{A}^{(1)}_{H,Q})^{-1} = D_H^{-\frac{1}{2}} (\bar{B} + \bar{V})^{-1}\, D_H^{-\frac{1}{2}}. \tag{4.31}$$

Furthermore, we observe that, as a consequence of Lemma 4.3, we have

$$\bar{B}_0 \geqslant \frac{2}{\vartheta\beta}\,\bar{P}, \qquad \text{where} \quad 1 < \frac{1}{\vartheta} := 1 + \frac{C_f}{4C_f} \leqslant 1 + \frac{\lambda_{\min}}{2C_f} \leqslant 2, \tag{4.32}$$

for $\alpha > 0$ sufficiently small and β sufficiently large. Note that we used $\lambda_{\min} \geqslant C_f/2$, see first paragraph of Sec. 3. We then have the following decay estimate.

Lemma 4.4. Assume Hypotheses 1, 2, and 3, and define $0 \leqslant \vartheta < 1$ by (4.32). Then

$$\|\{\bar{B}_0^{1/2} \bar{B}^{-1} \bar{B}_0^{1/2}\}_{ij}\| \leqslant \{(\mathbb{1} - \vartheta\alpha T)^{-1}\}_{ij}, \tag{4.33}$$

where $\|\cdot\|$ is the operator norm on $\mathscr{B}(\mathscr{H}^{(0)})$, see (2.44).

Proof. Expanding the inverse of $\bar{B}$ in a Neumann series, we obtain from (4.27) that

$$\bar{B}^{-1} := \sum_{n=0}^{\infty} \bar{B}_0^{-1} \left\{ \frac{2\alpha}{\beta}\, \bar{T} \bar{B}_0^{-1} \right\}^n \bar{P}. \tag{4.34}$$

Therefore, we have

$$\|\{\bar{B}_0^{1/2} \bar{B}^{-1} \bar{B}_0^{1/2}\}_{ij}\| \leqslant \sum_{n=0}^{\infty} \{M^n\}_{ij}, \tag{4.35}$$

where, cf. (4.32),

$$M_{ij} := \left\| \left\{ \bar{B}_0^{-1/2} \frac{2\alpha}{\beta} \bar{T} \bar{B}_0^{-1/2} \right\}_{ij} \right\| \leqslant \frac{2\alpha}{\beta} T_{ij} \, \|\bar{B}_0^{-1}\| \leqslant \frac{2\alpha}{\beta} T_{ij} \left(\frac{2}{\vartheta\beta} \right)^{-1} = \vartheta\alpha T_{ij}, \quad (4.36)$$

which, inserted into (4.35), yields the Neumann series for the inverse of the matrix $\mathbb{1} - \vartheta\alpha T$. ▮

Next, we note the following consequence of Theorem 2.9 and Lemma 4.1.

Lemma 4.5. Assume Hypotheses 1, 2, and 3. There exist universal constants $\alpha_0, \beta_0, C > 0$ such that, for all $0 \leqslant \alpha \leqslant \alpha_0$, $\beta \geqslant \beta_0$, and all $i, j \in \Lambda$, we have

$$\|\{\bar{B}_0^{-1/2} \bar{V} \bar{B}_0^{-1/2}\}_{ij}\| \leqslant \frac{C}{\beta^{1/2}} (\delta_{ij} + \alpha T_{ij}), \quad (4.37)$$

$$\|\{D_H^{-1/2} \bar{B}_0^{-1/2} \Delta_{\bar{P}P}\}_{ij} \, p\| \leqslant \frac{C}{\beta^{3/4}} (\delta_{ij} + \alpha T_{ij}). \quad (4.38)$$

Proof. We first remark that due to Theorem 2.9, there exist a universal constant C', such that

$$\|\{\bar{B}_0^{-1/2} \bar{V} \bar{B}_0^{-1/2}\}_{ij}\| \leqslant M_i \frac{C'}{\beta^{1/2}} (\delta_{ij} + \alpha a_{ij}) \, M_j, \quad (4.39)$$

$$\|\{\bar{B}_0^{-1/2} \bar{P} V P\}_{ij}\| \leqslant M_i \frac{C'}{\beta^{1/2}} (\delta_{ij} + \alpha a_{ij}) \, \|(\tilde{A}_j + \beta^{-1})^{1/2} \, p\|, \quad (4.40)$$

and

$$\|\{\bar{B}_0^{-1/2} \bar{P} B P\}_{ij}\| = \delta_{ij} \, \|(\bar{D}_j + 2\beta^{-1})^{-1/2} \, \bar{p} \tilde{A}_j \, p\| \leqslant \delta_{ij} M_j \, \|\tilde{A}_j^{1/2} p\|, \quad (4.41)$$

where

$$\begin{aligned} M_j &:= \|(\tilde{A}_j + \beta^{-1})^{1/2} \, \bar{p} (\bar{D}_j + 2\beta^{-1} \bar{p})^{-1/2}\| \\ &= \|(\bar{D}_j + 2\beta^{-1} \bar{p})^{-1/2} \, (\bar{p} \tilde{A}_j \bar{p} + \beta^{-1} \bar{p})(\bar{D}_j + 2\beta^{-1} \bar{p})^{-1/2}\|^{1/2} \\ &\leqslant C'', \end{aligned} \quad (4.42)$$

since $\tilde{A}_j \leqslant (C'')^2 D_j$, for some universal constant C''. Finally, we use Lemma 4.1 and arrive at

$$\|\tilde{A}_j^{1/2} p\| = \|p \tilde{A}_j p\|^{1/2} \leqslant \frac{C'''}{\beta^{3/4}}, \tag{4.43}$$

for some universal constant C'''. Since $T_{ij} = H''_{ii}(0)^{-1/2} a_{ij} H''_{jj}(0)^{-1/2}$ is bounded from above and below by universal multiples of a_{ij}, the lemma follows. ▮

We now come to proving the main result of this subsection: the fast decay of all terms but the main term in $\mathscr{F}_P$, defined in (4.2) and (4.3).

Theorem 4.6. Assume Hypotheses 1, 2, and 3, and let $\vartheta' \in (\vartheta, 1)$ be a universal number, where $\vartheta^{-1} := 1 + C_f/(4C_f)$ is defined in (4.32). Then there exist universal constants $\alpha_0, \beta_0, C > 0$ such that, for all $0 \leqslant \alpha \leqslant \alpha_0$, $\beta \geqslant \beta_0$, and all $i, j \in \Lambda$, we have

$$\|\bar{p} \{\bar{\Delta}_{H,Q}^{-1}\}_{ij}\, \bar{p}\| \leqslant C\beta \{(\mathbb{1} - \vartheta' \alpha T)^{-1}\}_{ij}, \tag{4.44}$$

$$\|\bar{p} \{\bar{\Delta}_{H,Q}^{-1}\, \Delta_{\bar{P}P}\}_{ij}\, p\| \leqslant \frac{C}{\beta^{1/4}} \{(\mathbb{1} - \vartheta' \alpha T)^{-1}\}_{ij}, \tag{4.45}$$

$$\|p\, \{\Delta_{\bar{P}P}^{*} \bar{\Delta}_{H,Q}^{-1} \Delta_{\bar{P}P}\}_{ij}\, p\| \leqslant \frac{C}{\beta^{3/2}} \{(\mathbb{1} - \vartheta' \alpha T)^{-1}\}_{ij}. \tag{4.46}$$

Proof. We only derive Estimate (4.46). The derivations of Estimates (4.44) and (4.45) are similar. We first observe that due to (4.31),

$$\{\Delta_{\bar{P}P}^{*} \bar{\Delta}_{H,Q}^{-1} \Delta_{\bar{P}P}\}_{ij} = \sum_{k,\ell} \{\Delta_{\bar{P}P}^{*} \bar{B}_0^{-1/2} D_H^{-\frac{1}{2}}\}_{ik} \tag{4.47}$$
$$\times \{(\bar{B}_0^{-1/2} \bar{B} \bar{B}_0^{-1/2} + \bar{B}_0^{-1/2} \bar{V} \bar{B}_0^{-1/2})^{-1}\}_{k\ell}\, \{D_H^{-\frac{1}{2}} \bar{B}_0^{-1/2}\, \Delta_{\bar{P}P}\}_{\ell j}.$$

A Neumann series expansion, (4.33), and (4.37) yield

$$\|\bar{p}\, \{(\bar{B}_0^{-1/2} \bar{B} \bar{B}_0^{-1/2} + \bar{B}_0^{-1/2} \bar{V} \bar{B}_0^{-1/2})^{-1}\}_{k\ell}\, \bar{p}\|$$
$$\leqslant \left\{ \sum_{n=0}^{\infty} (\mathbb{1} - \vartheta \alpha T)^{-1} \left[\frac{C}{\beta^{1/2}} (\mathbb{1} + \alpha T)(\mathbb{1} - \vartheta \alpha T)^{-1} \right]^n \right\}_{k\ell}. \tag{4.48}$$

Inserting this estimate and (4.38) into (4.47), we hence obtain

$$\begin{aligned}
&\|p\,\{\varDelta^*_{\bar P P}\,\bar\varDelta^{-1}_{H,Q}\,\varDelta_{\bar P P}\}_{ij}\,p\| \\
&\quad\leqslant \frac{C}{\beta^{1/2}}\left\{\sum_{n=1}^{\infty}\left[\frac{C}{\beta^{1/2}}(\mathbb{1}+\alpha T)(\mathbb{1}-\vartheta\alpha T)^{-1}\right]^n \frac{C}{\beta^{1/2}}(\mathbb{1}+\alpha T)\right\}_{ij} \\
&\quad\leqslant \frac{Cb^2}{\beta^{3/2}}\left\{\sum_{n=1}^{\infty}\left[\frac{C}{b}(\mathbb{1}+\alpha T)(\mathbb{1}-\vartheta\alpha T)^{-1}\right]^n \frac{C}{b}(\mathbb{1}+\alpha T)\right\}_{ij},
\end{aligned} \tag{4.49}$$

for any $b^2<\beta$, additionally using that all matrices involved have only nonnegative matrix elements. (At this point we would additionally make use of the trivial bound $\|\{D_H^{-1/2}\bar B_0^{-1/2}\}_{ij}\,\bar p\| \leqslant C\beta^{1/2}\delta_{ij}$ to derive (4.44) and (4.45).) We further observe that,

$$\delta_{ij} \leqslant \{(\mathbb{1}-\vartheta\alpha T)^{-1}\}_{ij}, \tag{4.50}$$

for all $i,j\in\Lambda$. Thus, for any matrix M with only nonnegative matrix elements $M_{ij}\geqslant 0$, we have that

$$M_{ij} \leqslant \{(\mathbb{1}-\vartheta\alpha T)^{-1}\,M(\mathbb{1}-\vartheta\alpha T)^{-1}\}_{ij}. \tag{4.51}$$

Applying this to the right side of (4.49), we arrive at

$$\begin{aligned}
\|p\{\varDelta^*_{\bar P P}\,\bar\varDelta^{-1}_{H,Q}\,\varDelta_{\bar P P}\}_{ij}\,p\| &\leqslant \frac{Cb^2}{\beta^{3/2}}\left\{\sum_{n=0}^{\infty}(\mathbb{1}-\vartheta\alpha T)^{-1}\left[\frac{C}{b}(\mathbb{1}+\alpha T)(\mathbb{1}-\vartheta\alpha T)^{-1}\right]^n\right\}_{ij} \\
&= \frac{Cb^2}{\beta^{3/2}}\{((1-Cb^{-1})\,\mathbb{1}-\alpha(\vartheta+Cb^{-1})\,T)^{-1}\}_{ij} \\
&= \frac{Cb^2}{(1-Cb^{-1})\,\beta^{3/2}}\left\{\left(\mathbb{1}-\alpha\frac{\vartheta+Cb^{-1}}{1-Cb^{-1}}T\right)^{-1}\right\}_{ij}.
\end{aligned} \tag{4.52}$$

The claim now follows from choosing a sufficiently large, universal number b such that $\vartheta+Cb^{-1}\leqslant(1-Cb^{-1})\,\vartheta'$. ∎

4.3. Proof of the Main Theorem

In this subsection we give a proof of our main result Theorem 1.1. This proof differs in an essential way from the corresponding proofs given in ref. 3 and 14, by not relying on translation invariance. We recall that translation invariance was used in order to diagonalize, using the Fourier transform, and then do estimates in momentum space.

We first present the correlation formula (2.24) with the resolvent of the Witten Laplacian expanded using the Feshbach reduction formula (4.4). Abbreviating

$$\mathscr{S}_P := \bar{P}(\bar{\Delta}^{(1)}_{H,Q})^{-1}\, \Delta_{\bar{P}P}\colon \operatorname{Ran} P \to \mathscr{H}^{(1)}, \tag{4.53}$$

$$\phi_j := \mathscr{Z}_\beta^{-1/2} e^{-\beta(H-q_j)} \otimes e_j, \tag{4.54}$$

for $j \in \Lambda$, we can write

$$\beta^2 \sqrt{H''_{ii}(0)\, H''_{jj}(0)}\; \mathbb{E}^T_\beta(x_i\,;x_j) \\ = \langle \phi_i \,|\, D_H^{\frac{1}{2}} \bar{P}(\bar{\Delta}^{(1)}_{H,Q})^{-1}\, \bar{P} D_H^{\frac{1}{2}} \phi_j \rangle + \langle \phi_i \,|\, D_H^{\frac{1}{2}} (P - \mathscr{S}_P)\, \mathscr{F}_P^{-1} (P - \mathscr{S}_P^*)\, D_H^{\frac{1}{2}}\, \phi_j \rangle. \tag{4.55}$$

Let U: Ran $P \to \mathbb{C}^\Lambda$ be given by

$$U\psi := \|e^{-\beta H}\|^{-1}_{\mathscr{H}^{(0)}} \sum_k \langle e^{-\beta H} \otimes e_k \,|\, \psi \rangle\, e_k. \tag{4.56}$$

Clearly $U^*U = \mathbb{1}_{\operatorname{Ran} P}$ and $UU^* = \mathbb{1}_{\mathbb{R}^\Lambda}$. After conjugation with this unitary map, the Feshbach operator $\mathscr{F}_P$ becomes a $\Lambda \times \Lambda$ matrix

$$\mathscr{F}_P =: \frac{2}{\beta} U^*\, D_H^{\frac{1}{2}} F D_H^{\frac{1}{2}} U, \tag{4.57}$$

where

$$F = \mathbb{1} - \alpha T - \beta^{-1/2} Y, \tag{4.58}$$

$$Y := \frac{\beta^{3/2}}{2} UP(D+V)\, PU^* - \frac{\beta^{3/2}}{2} UPD_H^{-\frac{1}{2}}\, \Delta^*_{\bar{P}P} (\bar{\Delta}^{(1)}_{H,Q})^{-1}\, \Delta_{\bar{P}P}\, D_H^{-\frac{1}{2}} PU^*. \tag{4.59}$$

To estimate the matrix elements of Y, we observe that Lemma 4.1 implies

$$\langle e_i \,|\, UPDPU^* e_j \rangle \leqslant C\delta_{ij}\, \|p\tilde{A}_j p\| \leqslant \frac{C'}{\beta^{3/2}}\, \delta_{ij}, \tag{4.60}$$

for some universal constants C and C'. Due to Theorem 2.9, we have that

$$\langle e_i \,|\, UPVPU^* e_j\rangle \leqslant \frac{C''}{\beta^{1/2}}\,(\delta_{ij}+\alpha T_{ij})\,\|(\tilde{A}_i+\beta^{-1})^{\frac{1}{2}}\,p\|\,\|(\tilde{A}_j+\beta^{-1})^{\frac{1}{2}}\,p\|$$
$$\leqslant \frac{C'''}{\beta^{3/2}}\,(\delta_{ij}+\alpha T_{ij}), \tag{4.61}$$

for some universal constants C'' and C'''. Estimates (4.60), (4.61), and (4.46) hence imply that there exist universal constants C and $\vartheta' \in (\vartheta, 1)$ such that

$$|Y_{ij}| \leqslant C\{(\mathbb{1}-\vartheta'\alpha T)^{-1}\}_{ij}. \tag{4.62}$$

Next, we introduce the $\Lambda \times \Lambda$ matrices ε, S, and $\bar{R}$ by

$$\varepsilon_{ij} := \delta_{ij}(\langle \phi_j \,|\, P\phi_j\rangle^{1/2}-1), \tag{4.63}$$
$$S_{ij} := \langle \phi_i \,|\, D_H^{\frac{1}{2}}\mathcal{S}_P\, D_H^{\frac{1}{2}} U^* e_j\rangle, \tag{4.64}$$
$$\bar{R}_{ij} := \langle \phi_i \,|\, D_H^{\frac{1}{2}}\bar{P}(\bar{\Delta}_{H,Q}^{(1)})^{-1}\,\bar{P}D_H^{\frac{1}{2}}\phi_j\rangle \tag{4.65}$$

and obtain

$$\frac{\beta}{2}\sqrt{H_{ii}''(0)\,H_{jj}''(0)}\;\mathbb{E}_\beta^T(x_i; x_j) = \{(\mathbb{1}+\varepsilon-S)\,F^{-1}(\mathbb{1}+\varepsilon^*-S^*)+\bar{R}\}_{ij}. \tag{4.66}$$

We observe that due to Corollary 2.4, there exists a universal $\delta > 0$ such that, for all $j \in \Lambda$,

$$|\langle \phi_j \,|\, \phi_j\rangle - 1| \leqslant e^{-2\,\delta\beta}, \tag{4.67}$$
$$|\langle \phi_j \,|\, P\phi_j\rangle - 1| \leqslant e^{-2\,\delta\beta}, \tag{4.68}$$
$$\langle \phi_j \,|\, \bar{P}\,\phi_j\rangle \leqslant e^{-2\,\delta\beta}, \tag{4.69}$$

provided $\alpha > 0$ is sufficiently small and β is sufficiently large. We recall from (4.32) that $\vartheta^{-1} := 1 + C_f/4C_f$, and we introduce a universal number $\vartheta' \in (\vartheta, 1)$ by $(\vartheta')^{-1} := 1 + C_f/8C_f$. Now, (4.44) and (4.45) of Theorem 4.6 (recall (4.53)) and (4.69) imply that there exists a universal C, such that for $\alpha > 0$ sufficiently small and β sufficiently large we have: For all $i, j \in \Lambda$,

$$|S_{ij}| \leqslant \frac{Ce^{-\delta\beta}}{\beta^{1/4}}\,\{(\mathbb{1}-\vartheta'\alpha T)^{-1}\}_{ij}, \tag{4.70}$$

$$|\bar{R}_{ij}| \leqslant C\beta e^{-2\,\delta\beta}\{(\mathbb{1}-\vartheta'\alpha T)^{-1}\}_{ij}. \tag{4.71}$$

Moreover, (4.68) directly yields that $|\varepsilon_{ij}| \leqslant \delta_{ij} e^{-\delta\beta}$. Thus, applying Theorems 3.3 and 3.4, using (4.62), we have

$$|\{(\varepsilon - S)\, F^{-1} + F^{-1}(\varepsilon^* - S^*) + (\varepsilon - S)\, F^{-1}(\varepsilon^* - S^*) + \bar{R}\}_{ij}| \leqslant e^{-\delta\beta}\{F^{-1}\}_{ij}, \tag{4.72}$$

for β sufficiently large, which, inserted into (4.66), yields

$$\begin{aligned}(1 - Ce^{-\delta\beta})\{F^{-1}\}_{ij} &\leqslant \frac{\beta}{2}\sqrt{H''_{ii}(0)\, H''_{jj}(0)}\; \mathbb{E}^T_\beta(x_i; x_j)\\ &\leqslant (1 + Ce^{-\delta\beta})\{F^{-1}\}_{ij},\end{aligned} \tag{4.73}$$

for all $i, j \in \Lambda$. Now applying again Theorem 3.3, we arrive at

$$\begin{aligned}&\left(1 - \frac{C'}{\sqrt{\beta}}\right)\left\{\left(\mathbb{1} - \frac{\alpha}{1 + C'\beta^{-1/2}}\, T\right)^{-1}\right\}_{ij}\\ &\leqslant \frac{\beta}{2}\sqrt{H''_{ii}(0)\, H''_{jj}(0)}\; \mathbb{E}^T_\beta(x_i\,; x_j)\\ &\leqslant \left(1 + \frac{C'}{\sqrt{\beta}}\right)\left\{\left(\mathbb{1} - \frac{\alpha}{1 - C'\beta^{-1/2}}\, T\right)^{-1}\right\}_{ij},\end{aligned} \tag{4.74}$$

where C' is universal. The first assertion (1.18) of Theorem 1.1 then results from an additional application of the Green's function estimates (3.32) and (3.33). The fact that ρ_H is a metric is proved in Theorem 3.8, and, finally, the second claim (1.19) is a transcription of (1.18). ▮

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